An algebraic geometric model of an action of the face monoid associated to a Kac-Moody group on its building

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Abstract

We described in [M1] a monoid $\widehat{G}$, the face monoid, acting on the integrable highest weight modules of a symmetrizable Kac-Moody algebra. It has similar structural properties as a reductive algebraic monoid whose unit group is a Kac-Moody group G. We found in [M5] two natural extensions of the action of the Kac-Moody group G on its building Ω to actions of the face monoid $\widehat{G}$ on the building Ω. Now we give an algebraic geometric model of one of these actions of the face monoid $\widehat{G}$ on the building Ω, where Ω is obtained as a part of the $\mathbb{F}$-valued points of the spectrum of homogeneous prime ideals of the Cartan algebra of the Kac-Moody group G. We determine all $\mathbb{F}$-valued points of the spectrum of homogeneous prime ideals of the Cartan algebra of G.

Introduction

The face monoid $\widehat{G}$ is an infinite-dimensional algebraic monoid. It has been obtained in [M1] by a Tannaka reconstruction from categories determined by the integrable highest weight representations of a symmetrizable Kac-Moody algebra. Compare also [M7], [M8]. By its construction and by the involved categories it is a very natural object. Its Zariski open dense unit group G coincides, up to a slightly extended maximal torus, with the special Kac-Moody group defined representation theoretically in [KP1].

The face monoid is a purely infinite-dimensional phenomenon, quite unexpected. In the classical case, i.e., if you take a split semisimple Lie algebra for the symmetrizable Kac-Moody algebra, it coincides with a split semisimple simply connected algebraic group. Put in another way, there seem to exist fundamentally different infinite-dimensional generalizations of a split semisimple simply connected algebraic group.

The results obtained in [M1], [M2], [M3] and [M4] show that the face monoid $\widehat{G}$ has similar structural and algebraic geometric properties as a reductive algebraic monoid, e.g., the monoid of $(n \times n)$-matrices. The face monoid $\widehat{G}$ is the first example of an infinite-dimensional reductive algebraic monoid. Actually, it is is particular. The investigation of the conjugagy classes in [M6] will show, that the relation between the face monoid $\widehat{G}$ and its unit group G, the Kac-Moody group, is much closer than for a general reductive algebraic monoid.

Obviously, there is the following question: Does the face monoid $\widehat{G}$ fit in some way to the bulding theory of the Kac-Moody group G? In [M5] we investigated how to extend the natural action of the Kac-Moody group G on its building Ω to actions of the face monoid $\widehat{G}$ on Ω. To explain the results obtained in [M5] note the following facts:

For the face monoid $\widehat{G}$ an infinite Renner monoid $\widehat{\mathcal{W}}$ plays the same role as the Weyl group $\mathcal{W}$ does for the Kac-Moody group G. For example, there are Bruhat and Birkhoff decompositions of $\widehat{G}$, similar as for G, but the Weyl group $\mathcal{W}$ replaced by the monoid $\widehat{\mathcal{W}}$. The monoid $\widehat{\mathcal{W}}$ can be constructed from the Weyl group $\mathcal{W}$ and the face lattice of the Tits cone X, where the term "face" means a face of the convex cone X in the sense of convex geometry. The Weyl group $\mathcal{W}$ is the unit group of $\widehat{\mathcal{W}}$.

The building Ω is covered by certain subcomplexes, the apartments, which are isomorphic to the Coxeter complex $\mathcal{C}$ associated to the Weyl group $\mathcal{W}$. This connects the action of G on Ω and the action of $\mathcal{W}$ on $\mathcal{C}$.

Now let $\mathcal{A}$ be the standard apartment associated to a fixed BN-pair (B, N) of the Kac-Moody group G. The group N may be obtained as normalizer $N = N_G(T)$ of the standard torus $T = B \cap N$, and the Weyl group $\mathcal{W}$ identifies with N/T. If we define similarly $\widehat{N} := N_{\widehat{G}}(T)$, then the monoid $\widehat{\mathcal{W}}$ identifies with $\widehat{N}/T$. Consider the following diagram:

$$\begin{array}{ccccc} \Omega & \supset & \mathcal{A} & \cong & \mathcal{C} \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ G & \supset & N & \twoheadrightarrow & \mathcal{W} \\ \cap & & \cap & & \cap \\ \widehat{G}\,? & \supset & \widehat{N} & \twoheadrightarrow & \widehat{\mathcal{W}} \end{array} \tag{1}$$

Suppose it is given an action of $\widehat{\mathcal{W}}$ on $\mathcal{C}$, extending the natural action of $\mathcal{W}$ on $\mathcal{C}$. Then this action induces uniquely an action of $\widehat{N}$ on $\mathcal{A}$, extending the natural action of N on $\mathcal{A}$. There are the following questions: Does there exist an action of $\widehat{G}$ on Ω, extending the natural action of G on Ω, such that diagram (1) commutes? If such an action exists, is it uniquely determined? Which actions can be obtained in this way?

In Theorem 45 and Corollary 48 of [M5] we obtained the following beautiful result: There is a bijective correspondence between:

(i) The actions of $\widehat{\mathcal{W}}$ on $\mathcal{C}$, extending the natural action of $\mathcal{W}$ on $\mathcal{C}$, satisfying certain conditions which we don't state here.

(ii) The actions of $\widehat{G}$ on Ω, extending the natural action of G on Ω.

Furthermore, the action of $\widehat{G}$ on Ω is obtained by an explicit formula from the action of $\widehat{\mathcal{W}}$ on $\mathcal{C}$.

There exist quite general actions of $\widehat{G}$ on Ω, extending the natural action of G on Ω. Compare for example the action of Remark 46 in [M5]. Is it possible to single out some actions of $\widehat{G}$ on Ω, which extend the natural action of G on Ω and also keep some of its properties? The natural action of G on the building Ω satisfies:

(a) It preserves the natural order on Ω.

(b) $\mathrm{Stab}_G(P) = P$ for all standard parabolic subgroups $P \in \Omega$.

Now every standard parabolic subgroup P of G extends naturally to a standard parabolic submonoid $\widehat{P}$ of $\widehat{G}$. An action of $\widehat{G}$ on Ω, which extends the natural action of G on Ω, is called good if it satisfies:

(a) It preserves the natural order on Ω.

(b) $\mathrm{Stab}_{\widehat{G}}(P) = \widehat{P}$ for all standard parabolic subgroups $P \in \Omega$.

It is open to determine all good actions. In [M5] we found two good actions of $\widehat{G}$, good action 1 and good action 2. Both have been obtained by the correspondence mentioned above from actions of $\widehat{\mathcal{W}}$ on $\mathcal{C}$, which in turn have been found by actions of $\widehat{\mathcal{W}}$ on the Tits cone and on Looijenga's modified Tis cone. It worked perfectly. But if you look at the formulas of these actions in Corollary 50 and Corollary 51 of [M5], you would like to have some natural models of these actions, if it is possible.

In this paper we give a natural algebraic geometric model of good action 1 of $\widehat{G}$. It is obtained as follows: Let P^+ be the set of dominant weights of the weight lattice. Let $L(\Lambda)$ be the irreducibel highest weight module of highest weight Λ, let $L(\Lambda)^{(*)}$ be its restricted dual. The Cartan algebra

$$CA = \bigoplus_{\Lambda \in P^+} L(\Lambda)^{(*)}$$

is a P^+-graded algebra over the field $\mathbb{F}$ of characteristic zero, whose multiplication on the graded parts

$$L(\Lambda_1)^{(*)} \otimes L(\Lambda_2)^{(*)} \to L(\Lambda_1 + \Lambda_2)^{(*)}$$

is obtained dually to G-equivariant embeddings $L(\Lambda_1 + \Lambda_2) \to L(\Lambda_1) \otimes L(\Lambda_2)$, $\Lambda_1, \Lambda_2 \in P^+$.

The spectrum $\mathrm{Proj}\,(CA)$ of all P^+-graded prime ideals, no irrelevant ideals excluded, is a locally ringed space. It is no scheme, but it is stratified by schemes. The building Ω identifies with a dense part of the $\mathbb{F}$-valued points $\mathrm{Proj}\,(CA)(\mathbb{F})$ of this spectrum, i.e.,

$$\Omega \hookrightarrow \mathrm{Proj}\,(CA)(\mathbb{F}).$$

A natural completion Ω_f of the building Ω identifies with the $\mathbb{F}$-valued points $\mathrm{Proj}\,(CA)(\mathbb{F})$ itself.

Now the face monoid $\widehat{G}$ acts on every module $L(\Lambda)$, $\Lambda \in P^+$. Dually we get actions of the opposite monoid $\widehat{G}^{op}$ on the restricted duals $L(\Lambda)^{(*)}$, $\Lambda \in P^+$, which fit together to an action of $\widehat{G}^{op}$ on the Cartan algebra CA by morphisms of graded algebras. This in turn induces actions of $\widehat{G}$ on $\mathrm{Proj}\,(CA)$ and $\mathrm{Proj}\,(CA)(\mathbb{F})$ by morphisms. The image of Ω in $\mathrm{Proj}\,(CA)(\mathbb{F})$ is invariant under the action of $\widehat{G}$. Pulling back the action of the face monoid $\widehat{G}$ to the building Ω gives good action 1.

A Cartan algebra can be obtained for every normal reductive algebraic monoid. It should be possible to investigate its spectrum of homogeneous prime ideals and the action of the reductive algebraic monoid on this spectrum similarly as in this article. Maybe also results of [M5] generalize in some way. As far as the author knows, actions of reductive algebraic monoids on the buildings of their unit groups or on building-like objects associated to their unit groups have not been investigated up to now. It may be interesting to do it.

1 Preliminaries

In this section we collect some facts about Kac-Moody algebras, minimal Kac-Moody groups, its associated face monoids, and formal Kac-Moody groups. It is a reference for the notation used in this article. It also indicates the representation theoretic construction of these groups and monoids used in the article.

We only state algebraic properties. We will also use some algebraic geometric properties of the Kac-Moody groups and their face monoids in the article. These are explained as soon as they are needed.

The facts on Kac-Moody algebras, which we state in this section, can be found in the books [K] (most results also valid for a field of characteristic zero with the same proofs), [Ku], and [Mo,Pi]. The facts on minimal Kac-Moody groups in [KP1], [KP3], [Ré], and [Mo,Pi]. The facts on formal Kac-Moody groups in [Ku], [Ré], and [Sl]. The facts on the face monoid in [M1] and [M5].

We denote by $\mathbb{N} = \mathbb{Z}^+$, $\mathbb{Q}^+$, resp. $\mathbb{R}^+$ the sets of strictly positive numbers of $\mathbb{Z}$, $\mathbb{Q}$, resp. $\mathbb{R}$, and the sets $\mathbb{N}_0 = \mathbb{Z}_0^+$, $\mathbb{Q}_0^+$, $\mathbb{R}_0^+$ contain, in addition, the zero. $\mathbb{F}$ is a field of characteristic 0 and $\mathbb{F}^\times$ its group of units.

The starting data: The starting data for the representation theoretic construction of the simply connected minimal and formal Kac-Moody groups and its face monoids are:

- A symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ with finite index set $I := \{1, 2, \ldots, n\}$. We denote by l the rank of A.
- A *simply connected minimal free root base* for A consisting of:
  - Dual free $\mathbb{Z}$-modules H, P of rank $2n - l$.
  - Linear independent sets $\{h_1, \ldots, h_n\} \subseteq H$, $\{\alpha_1, \ldots, \alpha_n\} \subseteq P$, such that $\alpha_i(h_j) = a_{ji}$, $i, j = 1, \ldots, n$.
  - Furthermore, there exist $\Lambda_1, \ldots, \Lambda_n \in P$ such that $\Lambda_i(h_j) = \delta_{ji}$, $i, j = 1, \ldots, n$.

P is called the *weight lattice*, $P^+ := \{\Lambda \in P \mid \Lambda(h_i) \geq 0 \text{ for all } i \in I\}$ the set of *dominant weights* of P. $Q := \mathbb{Z}\text{-span}\{\alpha_i \mid i \in I\}$ is called the *root lattice*. Set $Q_0^\pm := \mathbb{Z}_0^\pm\text{-span}\{\alpha_i \mid i \in I\}$ and $Q^\pm := Q_0^\pm \setminus \{0\}$. The height function $ht : Q \to \mathbb{Z}$ is defined by $ht(\sum_{i\in I} n_i\alpha_i) := \sum_{i\in I} n_i$.

$\{\alpha_i \mid i \in I\}$ is called the set of *simple roots*. Note: To cut short many formulas we often identify this set with the index set I.

We fix a system of elements $\Lambda_1, \ldots, \Lambda_n \in P$ such that $\Lambda_i(h_j) = \delta_{ji}$, $i, j = 1, \ldots, n$. We extend to dual bases $h_1, \ldots, \ldots, h_{2n-l} \in H$ and $\Lambda_1, \ldots, \Lambda_{2n-l} \in P$. We call $\Lambda_1, \ldots, \Lambda_{2n-l}$ a system of *fundamental dominant weights* of P.

Let $J \subseteq I$. Set $A_J := (a_{ij})_{i,j\in J}$, which is a symmetrizable generalized Cartan matrix if J is nonempty. Set $P_J := \mathbb{Z}\text{-span}\{\Lambda_i | i \in J\}$ and $P_J^+ := \mathbb{Z}_0^+\text{-span}\{\Lambda_i | i \in J\}$. Set $Q_J := \mathbb{Z}\text{-span}\{\alpha_i | i \in J\}$ and $(Q_J)_0^{\pm} := \mathbb{Z}_0^{\pm}\text{-span}\{\alpha_i | i \in J\}$ and $Q_J^{\pm} := (Q_J)_0^{\pm} \setminus \{0\}$. Set $P_{rest} := \mathbb{Z}\text{-span}\{\Lambda_i \,|\, i = n+1, \ldots, 2n-l\}$. As always, a span of the empty set is defined to be $\{0\}$.

The linear spaces h and h*: Define the $\mathbb{F}$-linear spaces

$$\mathbf{h} := \mathbf{h}_{\mathbb{F}} := H \otimes_{\mathbb{Z}} \mathbb{F} \quad \text{and} \quad \mathbf{h}^* = \mathbf{h}_{\mathbb{F}}^* := P \otimes_{\mathbb{Z}} \mathbb{F}.$$

Identify H and P with the corresponding sublattices of $\mathbf{h}$ and $\mathbf{h}^*$. Interprete $\mathbf{h}^*$ as the dual of $\mathbf{h}$. Order the elements of $\mathbf{h}^*$ by $\lambda \le \lambda'$ if and only if $\lambda' - \lambda \in Q_0^+$.

For $J \subseteq I$ set $\mathbf{h}_J := \text{span}\{h_i | i \in J\}$. Set $\mathbf{h}_{rest} := \text{span}\{h_i | i = n+1, \ldots, 2n-l\}$. In the same way as in §2.1 of [K] equip $\mathbf{h}$ with a non-degenerate symmetric bilinear form $(\,|\,)$ adapted to the decomposition $\mathbf{h} = \mathbf{h}_I \oplus \mathbf{h}_{rest}$. Denote by $\nu : \mathbf{h} \to \mathbf{h}^*$ be the induced linear isomorphism. Denote also by $(\,|\,)$ the induced non-degenerate symmetric bilinear form on $\mathbf{h}^*$.

The Weyl group $\mathcal{W}$: The *Weyl group* $\mathcal{W} = \mathcal{W}(A)$ is the Coxeter group with generators σ_i, $i \in I$, and relations

$$\sigma_i^2 = 1 \qquad (i \in I) \qquad , \qquad (\sigma_i\sigma_j)^{m_{ij}} = 1 \qquad (i, j \in I,\, i \ne j),$$

where the m_{ij} are given by:

$a_{ij}a_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	no relation between σ_i and σ_j

Denote by $l : \mathcal{W} \to \mathbb{N}_0$ the *length function*. Denote by $\mathcal{W}_J \cong \mathcal{W}(A_J)$, $J \subseteq I$, the *standard parabolic subgroups* of $\mathcal{W}$. For $J, K \subseteq I$ denote by $\mathcal{W}^J$ the set of minimal coset representatives of $\mathcal{W}/\mathcal{W}_J$, by ${}^K\mathcal{W}$ the set of minimal coset representatives of $\mathcal{W}_K\backslash\mathcal{W}$, and by ${}^K\mathcal{W}^J$ the set of minimal double coset representatives of $\mathcal{W}_K\backslash\mathcal{W}/\mathcal{W}_J$.

The Tits cone X: The Weyl group $\mathcal{W}$ acts faithfully $\mathbf{h}^*$ by

$$\sigma_i\lambda := \lambda - \lambda(h_i)\alpha_i \qquad i \in I, \quad \lambda \in \mathbf{h}^*.$$

The lattices Q and P are left invariant by this action.

The *Tits cone* is a convex $\mathcal{W}$-invariant cone which may be obtained by

$$X := \mathcal{W}\overline{C} \quad \text{with} \quad \overline{C} := \{\lambda \in \mathbf{h}_{\mathbb{R}}^* | \lambda(h_i) \ge 0 \text{ for all } i \in I\}.$$

Every $\mathcal{W}$-orbit of X contains exactly one point of the (closed) *standard fundamental chamber* $\overline{C}$. Now we recall shortly:

1. A $\mathcal{W}$-invariant partition of X given by the interiors of polyhedral cones, which we call the facets of X.

2. The set $Fa(X)$ of faces of X in the sense of convex geometry. It is a $\mathcal{W}$-space, partially ordered by the inclusion of sets, even a complete lattice.

1. For $J \subseteq I$ set

$$\begin{aligned} F_J &:= \{\lambda \in \mathbf{h}_{\mathbb{R}}^* | \lambda(h_i) = 0 \text{ for } i \in J, \quad \lambda(h_i) > 0 \text{ for } i \in I \setminus J\}, \\ \overline{F_J} &:= \{\lambda \in \mathbf{h}_{\mathbb{R}}^* | \lambda(h_i) = 0 \text{ for } i \in J, \quad \lambda(h_i) \ge 0 \text{ for } i \in I \setminus J\} = \dot{\bigcup_{K \supseteq J}} F_K. \end{aligned}$$

In particular, $\overline{C} = \overline{F_\emptyset}$. Here $\overline{F_J}$ is a polyhedral cone with relative interior F_J, its faces given by $\overline{F_K}$, $K \supseteq J$. We call F_J resp. $\overline{F_J}$ an open resp. closed *standard facet* of type J.

The parabolic subgroup $\mathcal{W}_J$ of $\mathcal{W}$ is the stabilizer of any element $\lambda \in F_J$. It is the stabilizer of F_J as well as of $\overline{F_J}$ as a whole.

The set $\{\sigma F_J | \sigma \in \mathcal{W}, J \subseteq I\}$ is a $\mathcal{W}$-invariant partition of X. Partially ordered by the reverse closure relation it gives the coxeter complex of $\mathcal{W}$. We call σF_J resp. $\sigma\overline{F_J}$ an open resp. closed *facet* of type J.

2. For $\emptyset \neq \Theta \subseteq I$ we denote by Θ^0, resp. Θ^∞ the set of indices which correspond to the sum of the components of the generalized Cartan submatrix A_Θ of finite, resp. non-finite type. We set $\emptyset^0 := \emptyset^\infty := \emptyset$. For $J \subseteq I$ set $J^\perp := \{ i \in I | a_{ij} = 0 \text{ for all } j \in J\}$.

A set $\Theta \subseteq I$ such that $\Theta = \Theta^\infty$ is called *special*. If Θ is a special set then

$$R(\Theta) := \mathcal{W}_{\Theta^\perp}\overline{F_\Theta} = \{ \lambda \in X | \lambda(h_i) = 0 \text{ for all } i \in \Theta\}$$

is a face of X with relative interior

$$ri\,(R(\Theta)) = \mathcal{W}_{\Theta^\perp} \bigcup_{J \subseteq \Theta^\perp,\, J = J^0} F_{\Theta \cup J}.$$

The $\mathcal{W}$-stabilizers of $R(\Theta)$, pointwise and as a whole, are

$$Z_\mathcal{W}(R(\Theta)) = \mathcal{W}_\Theta \quad \text{and} \quad N_\mathcal{W}(R(\Theta)) = \mathcal{W}_{\Theta \cup \Theta^\perp}.$$

It holds $\{ R(\Theta) | \Theta \textit{ special} \} = \left\{ R \in \mathrm{Fa}(X) | ri(R) \cap \overline{C} \neq \emptyset \right\}$. Furthermore, it holds

$$Fa(X) = \dot{\bigcup_{\Theta \textit{ special}}} \{\sigma R(\Theta) | \sigma \in \mathcal{W}\}.$$

The special set Θ is called the *type* of the face $\sigma R(\Theta)$.

Let $\sigma, \sigma' \in \mathcal{W}$, Θ, Θ' be special. It holds

$$\sigma' R(\Theta') \subseteq \sigma R(\Theta) \quad \Longleftrightarrow \quad \Theta' \supseteq \Theta \text{ and } \sigma^{-1}\sigma' \in \mathcal{W}_{\Theta^\perp}\mathcal{W}_{\Theta'}.$$

Different faces of X of the same type are not comparable by $\subseteq$.

A well defined function $red : \mathcal{W} \to I$ is obtained as follows: Set $red\,(1) := \emptyset$. If $\sigma \in \mathcal{W} \setminus \{1\}$ and $\sigma_{i_1} \cdots \sigma_{i_k}$ is a reduced expression for σ set $red\,(\sigma) := \{i_1, \ldots, i_k\} \subseteq I$.

The lattice intersection, which coincides with the set theoretical intersection, and the lattice join of two arbitrary faces can be reduced easily by the formulas for the stabilizers to the following formulas: Let $\tau \in {}^{\Theta_1 \cup \Theta_1^\perp}\mathcal{W}^{\Theta_2 \cup \Theta_2^\perp}$, Θ_1, Θ_2 be special. It holds

$$\begin{aligned} R(\Theta_1) \cap \tau R(\Theta_2) &= R(\Theta_1 \cup \Theta_2 \cup red\,(\tau)), \\ R(\Theta_1) \vee \tau R(\Theta_2) &= R((\Theta_1 \cap \tau\Theta_2)^\infty). \end{aligned}$$

The Kac-Moody algebra g: The *Kac-Moody algebra* $\mathbf{g} = \mathbf{g}(A)$ is the Lie algebra over $\mathbb{F}$ generated by the abelian Lie algebra $\mathbf{h}$, and the elements e_i, f_i, $(i \in I)$, with the following relations, which hold for any $i, j \in I$, $h \in \mathbf{h}$:

$$\begin{gathered} [e_i, f_j] = \delta_{ij} h_i \;, \; [h, e_i] = \alpha_i(h) e_i \;, \; [h, f_i] = -\alpha_i(h) f_i \;, \\ (ad\, e_i)^{1-a_{ij}}\, e_j = (ad\, f_i)^{1-a_{ij}}\, f_j = 0 \qquad (i \neq j) \;. \end{gathered}$$

The abelian Lie algebra $\mathbf{h}$ and the elements e_i, f_i, $(i \in I)$, identify with their images in $\mathbf{g}$.

There is the *root space decomposition*

$$\mathbf{g} = \bigoplus_{\alpha \in \mathbf{h}^*} \mathbf{g}_\alpha \quad \text{where} \quad \mathbf{g}_\alpha := \{ x \in \mathbf{g} | [h, x] = \alpha(h)\, x \text{ for all } h \in \mathbf{h}\}.$$

In particular, it holds $\mathbf{g}_0 = \mathbf{h}$, $\mathbf{g}_{\alpha_i} = \mathbb{F}e_i$, and $\mathbf{g}_{-\alpha_i} = \mathbb{F}f_i$, $i \in I$. The set of *roots* $\Delta := \{ \alpha \in \mathbf{h}^* \setminus \{0\} | \mathbf{g}_\alpha \neq 0\}$ satisfies $\Delta \subseteq Q$, $\Delta = -\Delta$, and it is invariant under the Weyl group. $\Delta_{re} :=$

$\mathcal{W}\{\alpha_i | i \in I\}$ is the set of *real roots*. $\Delta_{im} := \Delta \setminus \Delta_{re}$ is the set of *imaginary roots*. Every root space $\mathbf{g}_\alpha$, $\alpha \in \Delta$, is finite dimensional.

Δ, Δ_{re}, and Δ_{im} decompose into the disjoint union of the sets of *positive* and *negative roots* $\Delta^\pm := \Delta \cap Q^\pm$, ${\Delta_{re}}^\pm := \Delta_{re} \cap Q^\pm$, ${\Delta_{im}}^\pm := \Delta_{im} \cap Q^\pm$. There is the corresponding *triangular decomposition*

$$\mathbf{g} = \mathbf{n}^- \oplus \mathbf{h} \oplus \mathbf{n}^+ \quad \text{where} \quad \mathbf{n}^\pm := \bigoplus_{\alpha \in \Delta^\pm} \mathbf{g}_\alpha \,.$$

For every $\alpha \in \Delta_{re}$ the subalgebra $\mathbf{g}_\alpha \oplus [\mathbf{g}_\alpha, \mathbf{g}_{-\alpha}] \oplus \mathbf{g}_{-\alpha}$ of $\mathbf{g}$ is isomorphic to $sl(2, \mathbb{F})$.

The non-degenerate symmetric bilinear form $(\ |\ )$ on $\mathbf{h}$ extends uniquely to a non-degenerate symmetric invariant bilinear form $(\ |\ )$ on $\mathbf{g}$. For $\alpha, \beta \in \Delta \cup \{0\}$ such that $\alpha + \beta \neq 0$ it holds $(\mathbf{g}_\alpha | \mathbf{g}_\beta) = 0$ The restriction $(\ |\ ) : \mathbf{g}_\alpha \times \mathbf{g}_{-\alpha} \to \mathbb{F}$ is nondegenerate, and $[x, y] = (x|y)\nu^{-1}(\alpha)$ for all $x \in \mathbf{g}_\alpha$, $y \in \mathbf{g}_{-\alpha}$, $\alpha \in \Delta$.

Let $J \subseteq I$. Set

$$\mathbf{g}_J := \mathbf{h}_J \oplus \bigoplus_{\alpha \in \Delta_J} \mathbf{g}_\alpha = \mathbf{n}_J^- \oplus \mathbf{h}_J \oplus \mathbf{n}_J^+ \quad \text{where} \quad \mathbf{n}_J^\pm := \bigoplus_{\alpha \in \Delta_J^\pm} \mathbf{g}_\alpha$$

and $\Delta_J := \Delta \cap Q_J$, $\Delta_J^\pm := \Delta^\pm \cap Q_J$. Then $\mathbf{g}_J \cong \mathbf{g}(A_J)'$ for $J \neq \emptyset$. In particular, $\mathbf{g}_I$ is the derived Lie algebra of $\mathbf{g}$.

An ideal of $\mathbf{n}^\pm$ is obtained by

$$(\mathbf{n}^\pm)^J := \bigoplus_{\alpha \in (\Delta^\pm)^J} \mathbf{g}_\alpha \quad \text{with} \quad (\Delta^\pm)^J := \Delta^\pm \setminus \Delta_J^\pm .$$

The category $\mathcal{O}^p_{int}$: The category $\mathcal{O}^p$ is defined as follows: Its objects are the $\mathbf{g}$-modules V, which have the properties:

1. V is $\mathbf{h}$-diagonalizable with finite dimensional weight spaces.
2. There exist finitely many elements $\lambda_1, \ldots, \lambda_m \in \mathbf{h}^*$, such that the set of weights $P(V)$ of V is contained in the union $\bigcup_{1=1}^m \{\lambda \in \mathbf{h}^* | \lambda \leq \lambda_i\}$.

The morphisms of $\mathcal{O}^p$ are the morphisms of $\mathbf{g}$-modules. The category $\mathcal{O}^n$ is defined similarly, but $\leq$ is replaced by $\geq$ in the second condition.

Call a $\mathbf{g}$-module V *integrable* if V is $\mathbf{h}$-diagonalizable with set of weights $P(V) \subseteq P$, and the elements of $\mathbf{g}_\alpha$ act locally nilpotent on V for all $\alpha \in \Delta_{re}$. Examples of integrable representations are the adjoint representation $\mathbf{g}$, and the irreducible highest weight representations $L(\Lambda)$, $\Lambda \in P^+$. We denote the set of weights of $L(\Lambda)$ by $P(\Lambda)$, $\Lambda \in P^+$.

Let $\mathcal{O}^p_{int}$ the full subcategory of the category $\mathcal{O}^p$, whose objects are integrable modules. This category generalizes the category of finite dimensional representations of a semisimple Lie algebra, keeping the complete reducibility theorem: Every object of $\mathcal{O}^p_{int}$ is isomorphic to a direct sum of the integrable irreducible highest weight modules $L(\Lambda)$, $\Lambda \in P^+$. In particular, the set of weights of every object of $\mathcal{O}^p_{int}$ is contained in $X \cap P$ because of $\bigcup_{\Lambda \in P^+} P(\Lambda) = X \cap P$.

Let $Nat(\mathcal{O}^p_{int})$ be the set of natural transformations of the forgetful functor from the category $\mathcal{O}^p_{int}$ into the category of $\mathbb{F}$-linear spaces. Explicitly, $Nat(\mathcal{O}^p_{int})$ consists of the families of linear maps

$$m \quad = \quad (\, m_V \in End(V)\,)_{V \; obj. \; of \; \mathcal{O}^p_{int}} \;,$$

such that for all objects V, W, and all morphisms $\phi : V \to W$ of $\mathcal{O}^p_{int}$ the diagram

$$\begin{array}{ccc} V & \stackrel{m_V}{\to} & V \\ \phi \downarrow & & \downarrow \phi \\ W & \stackrel{m_W}{\to} & W \end{array}$$

commutes. $Nat(\mathcal{O}^p_{int})$ gets in the obvious way the structure of an associative $\mathbb{F}$-algebra with unit.

Note: Let V be an object of $\mathcal{O}_{int}^p$ and $V = \bigoplus_{\lambda \in P(V)} V_\lambda$ its weight space decomposition. We may associate two $\mathbb{F}$-linear duals, the restricted dual $V^{(*)}$ and the full dual V^* as follows

$$V^{(*)} := \bigoplus_{\lambda \in P(V)} V_\lambda^* \quad \subseteq \quad V^* = \prod_{\lambda \in P(V)} V_\lambda^* .$$

Note also: For everything which follows it would be possible to use any category generated in some sense by the integrable highest weight modules. The category $\mathcal{O}_{int}^p$ is used, because it is easy to work with.

The face monoid $\widehat{G}$ associated to the (minimal) Kac-Moody group G : Consider the following elements of $Nat(\mathcal{O}_{int}^p)$:

(1) For every $h \in H$, $s \in \mathbb{F}^\times$ there exists $t_h(s) \in Nat(\mathcal{O}_{int}^p)$, such that for every object V of $\mathcal{O}_{int}^p$ it holds

$$t_h(s) v_\lambda = s^{\lambda(h)} v_\lambda \quad , \quad v_\lambda \in V_\lambda \ , \ \lambda \in P(V).$$

(2) For every $x \in \mathbf{g}_\alpha$, $\alpha \in \Delta_{re}$, there exists $exp(x) \in Nat(\mathcal{O}_{int}^p)$, such that for every object V of $\mathcal{O}_{int}^p$ it holds

$$exp(x) v = \sum_{k \in \mathbb{N}_0} \frac{x^k v}{k!} \quad , \quad v \in V.$$

(3) For every $R \in Fa(X)$ there exists $e(R) \in Nat(\mathcal{O}_{int}^p)$, such that for every object V of $\mathcal{O}_{int}^p$ it holds

$$e(R) v_\lambda = \begin{cases} v_\lambda & \lambda \in R \\ 0 & \lambda \in X \setminus R \end{cases} \quad , \quad v_\lambda \in V_\lambda \ , \ \ \lambda \in P(V).$$

The *face monoid* $\widehat{G}$ is the submonoid of $Nat(\mathcal{O}_{int}^p)$ generated by the elements of (1), (2), and (3). It is the biggest part of $Nat(\mathcal{O}_{int}^p)$, whose elements are compatible with $\oplus$, $\otimes$, act as identity on the trivial representation, and induce opposite actions on the restricted duals.

The unit group G of $\widehat{G}$ is generated by the elements of (1) and (2). It is isomorphic to the *minimal Kac-Moody group*, which we call *Kac-Moody group* for short. It acts on $\mathbf{g}$ by the adjoint representation. It has the following important structural properties:

A root group data system $(T, (U_\alpha)_{\alpha \in \Delta_{re}})$ of G is obtained as follows: The elements of (1) induce an embedding of the torus $H \otimes_{\mathbb{Z}} \mathbb{F}^\times = Hom((P,+), (\mathbb{F}^\times, \cdot))$ into G, whose image is T. For every $\alpha \in \Delta_{re}$ the elements of (2) induce an embedding of $(\mathbf{g}_\alpha, +)$ into G, whose image is the *root group* U_α.

The group $N := N_G(T)$ is generated by T and the elements $n_i := exp(e_i) exp(-f_i) exp(e_i)$, $i \in I$. An isomorphism $\kappa : \mathcal{W} \to N/T$ is induced by $\kappa(\sigma_i) := n_i T$, $i \in I$.

We denote an arbitrary element $n \in N$ with $\kappa^{-1}(nT) = \sigma \in \mathcal{W}$ by n_σ. The set of weights $P(V)$ of an integrable $\mathbf{g}$-module V is $\mathcal{W}$-invariant, and it holds $n_\sigma V_\lambda = V_{\sigma\lambda}$, $\lambda \in P(V)$, $\sigma \in \mathcal{W}$.

We denote by $\widetilde{\ \ } : \mathcal{W} \to N$ the cross section of the canonical map from N to $\mathcal{W}$ defined by $\widetilde{1} = 1$, $\widetilde{\sigma_i} = n_i$, $i \in I$, and $\widetilde{\sigma\tau} = \widetilde{\sigma}\widetilde{\tau}$ if $l(\sigma\tau) = l(\sigma) + l(\tau)$, $\sigma, \tau \in \mathcal{W}$.

Let $U^\pm$ be the subgroups of G generated by U_α, $\alpha \in \Delta_{re}^\pm$. Then $U^\pm$ are normalized by T. The pairs $(B^\pm := T \ltimes U^\pm, N)$ are twin BN-pairs of G with the property $B^+ \cap B^- = B^\pm \cap N = T$. There are the *Bruhat* and *Birkhoff decompositions*

$$G = \dot{\bigcup_{\sigma \in N}} U^\epsilon n U^\delta = \dot{\bigcup_{\sigma \in \mathcal{W}}} B^\epsilon \sigma B^\delta \qquad , \qquad \epsilon, \delta \ \in \ \{+, -\}.$$

There are also Levi decompositions of the standard parabolic subgroups $P_J^\pm$, $J \subseteq I$, and for $U^\pm$:

$$P_J^\pm = L_J^\pm \ltimes (U^\pm)^J \quad \text{ and } \quad U^\pm = U_J^\pm \ltimes (U^\pm)^J.$$

Here $U_J^\pm$ is the group generated by U_α, $\alpha \in (\Delta_J^\pm)_{re} := \Delta_J^\pm \cap \Delta_{re}$. $(U^\pm)^J$ is the smallest normal subgroup of $U^\pm$ containing U_α, $\alpha \in (\Delta^\pm)_{re}^J := (\Delta^\pm)^J \cap \Delta_{re}$. This group equals $\bigcap_{\sigma \in \mathcal{W}_J} \sigma U^\pm \sigma^{-1}$. Furtermore, $L_J^\pm := U_J^\pm (T\mathcal{W}_J) U_J^\pm$.

For $J \subseteq I$ let G_J be the subgroup of G generated by U_α, $\alpha \in (\Delta_J)_{re} := \Delta_J \cap \Delta_{re}$. Then $G_J \cong G(A_J)'$ for $J \neq \emptyset$. The torus $T_J := G_J \cap T$ is generated by the subtori $t_{h_i}(\mathbb{F}^\times)$, $i \in J$. The group $N_J := G_J \cap N$ is generated by T_J and the elements n_i, $i \in J$.

In particular, G_I is the derived group of G, which identifies with the Kac-Moody group as defined in [KP1]. It holds $G = G_I \rtimes T_{rest}$, where T_{rest} is the subtorus of T generated by the subtori $t_{h_i}(\mathbb{F}^\times)$, $i = n+1, \ldots, 2n-l$.

For $J \subseteq I$ let T^J be the subtorus of T generated by $t_{h_i}(\mathbb{F}^\times)$, $i \in \{1, 2, \ldots, 2n-l\} \setminus J$.

Let Θ be special. There are the decompositions of the parabolic subgroups

$$\begin{aligned} P_{\Theta\cup\Theta^\perp} &= \left(G_{\Theta^\perp} \rtimes T^{\Theta\cup\Theta^\perp}\right) \ltimes \left(G_\Theta \ltimes U^{\Theta\cup\Theta^\perp}\right), \\ P^-_{\Theta\cup\Theta^\perp} &= \left((U^-)^{\Theta\cup\Theta^\perp} \rtimes G_\Theta\right) \rtimes \left(G_{\Theta^\perp} \rtimes T^{\Theta\cup\Theta^\perp}\right). \end{aligned}$$

The projections belonging to these semidirect products are denoted by

$$p^\pm_\Theta \quad : \quad P^\pm_{\Theta\cup\Theta^\perp} \to G_{\Theta^\perp} \rtimes T^{\Theta\cup\Theta^\perp}.$$

Now the monoid $\widehat{G}$ may be described algebraically as follows: It holds

$$\widehat{G} = \dot{\bigcup_{\Theta \ special}} G e(R(\Theta)) G.$$

Let $g_1, g_2, h_1, h_2 \in G$, let Θ_1, Θ_2 be special. It holds $g_1 e(R(\Theta_1))h_1 = g_2 e(R(\Theta_2))h_2$ if and only if $\Theta_2 = \Theta_1$ and

$$(g_2)^{-1}g_1 \in P_{\Theta_1\cup\Theta_1^\perp}, \;\; h_2(h_1)^{-1} \in P^-_{\Theta_1\cup\Theta_1^\perp} \;\text{ with }\; p_{\Theta_1}((g_2)^{-1}g_1) = p^-_{\Theta_1}(h_2(h_1)^{-1}).$$

To describe the multiplication of $g_1 e(R(\Theta_1))h_1$ and $g_2 e(R(\Theta_2))h_2$ write $h_1 g_2 = a_1 \widetilde{\sigma} a_2$ with $a_1 \in P^-_{\Theta_1\cup\Theta_1^\perp}$, $a_2 \in P_{\Theta_2\cup\Theta_2^\perp}$, and $\sigma \in {}^{\Theta_1\cup\Theta_1^\perp}\mathcal{W}^{\Theta_2\cup\Theta_2^\perp}$. It holds

$$g_1 e(R(\Theta_1))h_1\, g_2 e(R(\Theta_2))h_2 = g_1 p^-_{\Theta_1}(a_1)\, e(\, R(\Theta_1 \cup \Theta_2 \cup red\,(\sigma))\,)\, p_{\Theta_2}(a_2) h_2.$$

The monoid $\widehat{N} := N_{\widehat{G}}(T) := \left\{ \hat{g} \in \widehat{G} \,\middle|\, \hat{g}T = T\hat{g} \right\}$ is generated by N and $\{\, e(R) |\, R \in \mathrm{Fa}(X)\}$. There are the Bruhat and Birkhoff decompositions

$$\widehat{G} = \dot{\bigcup_{\widehat{n}\,\in\,\widehat{N}}} U^\epsilon \widehat{n} U^\delta = \dot{\bigcup_{\widehat{w}\,\in\,\widehat{N}/T}} B^\epsilon \widehat{w} B^\delta \qquad , \qquad \epsilon, \delta \,\in\, \{\,+\,,\,-\,\}.$$

The Weyl group $\mathcal{W}$ acts on the monoid $(\,\mathrm{Fa}(X)\,,\,\cap\,)$. The semidirect product $\mathcal{W} \ltimes \mathrm{Fa}(X)$ consists of the set $\mathcal{W} \times \mathrm{Fa}(X)$ on which the structure of a monoid is given by

$$(\sigma, R) \cdot (\tau, S) := (\sigma\tau, \tau^{-1}R \cap S).$$

A congruence relation on $\mathcal{W} \ltimes \mathrm{Fa}(X)$ is obatined by

$$(\sigma, R) \sim (\sigma', R') \quad :\Longleftrightarrow \quad R \,=\, R' \quad \text{and} \quad \sigma^{-1}\sigma' \in Z_{\mathcal{W}}(R)\,.$$

We denote the congruence class of (σ, R) by $\sigma\varepsilon(R)$. Now the monoid $\widehat{N}/T$ is isomorphic to the monoid $\widehat{\mathcal{W}} := (\mathcal{W} \ltimes \mathrm{Fa}(X))/\sim$, which we call the *face monoid* associated to $\mathcal{W}$, by

$$\kappa : \begin{array}{ccc} \widehat{\mathcal{W}} & \to & \widehat{N}/T \\ \sigma\varepsilon(R) & \mapsto & \widetilde{\sigma} e(R) T \end{array}.$$

The Weyl group $\mathcal{W}$ is the unit group of $\widehat{\mathcal{W}}$. There are naturally standard parabolic submonoids $\widehat{\mathcal{W}}_J$, $J \subseteq I$, of $\widehat{\mathcal{W}}$. These give the standard parabolic submonoids $\hat{P}_J := B\widehat{\mathcal{W}}_J B$, $J \subseteq I$, of $\widehat{G}$.

The formal Kac-Moody algebras $\mathbf{g}_{fp}$, $\mathbf{g}_{fn}$: Set

$$\begin{aligned} \mathbf{n}_f := \mathbf{n}_f^+ := \textstyle\prod_{\alpha\in\Delta^+} \mathbf{g}_\alpha \quad &\text{and} \quad \mathbf{g}_{fp} := \mathbf{n}^- \oplus \mathbf{h} \oplus \mathbf{n}_f, \\ \mathbf{n}_f^- := \textstyle\prod_{\alpha\in\Delta^-} \mathbf{g}_\alpha \quad &\text{and} \quad \mathbf{g}_{fn} := \mathbf{n}_f^- \oplus \mathbf{h} \oplus \mathbf{n}. \end{aligned}$$

The Lie bracket of $\mathbf{g}$ extends in the obvious way to Lie brackets of $\mathbf{g}_{fp}$, $\mathbf{g}_{fn}$.

For $J \subseteq I$ there are the semidirect decompositions

$$\mathbf{n}_f^\pm = (\mathbf{n}_f^\pm)_J \ltimes (\mathbf{n}_f^\pm)^J \quad \text{with} \quad (\mathbf{n}_f^\pm)_J := \prod_{\alpha\in\Delta_J^\pm} \mathbf{g}_\alpha \ \text{ and } \ (\mathbf{n}_f^\pm)^J := \prod_{\alpha\in(\Delta^\pm)^J} \mathbf{g}_\alpha .$$

The formal Kac-Moody group G_{fp}: For every object V of $\mathcal{O}^p_{int}$ the action of the Kac-Moody algebra $\mathbf{g}$ on V extends in the obviuos way to an action of the formal Kac-Moody algebra $\mathbf{g}_{fp}$ on V. Consider the following elements of $Nat(\mathcal{O}^p_{int})$:

(1) For every $h \in H$, $s \in \mathbb{F}^\times$ there exists $t_h(s) \in Nat(\mathcal{O}^p_{int})$, such that for every object V of $\mathcal{O}^p_{int}$ it holds

$$t_h(s)v_\lambda = s^{\lambda(h)}v_\lambda \quad , \quad v_\lambda \in V_\lambda \ , \ \lambda \in P(V).$$

(2) For every $x \in \mathbf{n}_f \cup \bigcup_{\alpha\in\Delta^-_{re}} \mathbf{g}_\alpha$ there exists $exp(x) \in Nat(\mathcal{O}^p_{int})$, such that for every object V of $\mathcal{O}^p_{int}$ it holds

$$exp(x)v = \sum_{k\in\mathbb{N}_0} \frac{x^k v}{k!} \quad , \quad v \in V.$$

The *formal Kac-Moody group* G_{fp} is the submonoid of $Nat(\mathcal{O}^p_{int})$ generated by the elements of (1) and (2). Its elements are compatible with $\oplus$, $\otimes$, act as identity on the trivial representation, and induce opposite actions on the full duals. G_{fp} acts on $\mathbf{g}_{fp}$ by the adjoint representation.

The formal Kac-Moody group G_{fp} contains the (minimal) Kac-Moody group G. It has the following important structural properties:

$U_f := exp(\mathbf{n}_f)$ is a prounipotent group. It holds $G_{fp} = U_f G = G U_f$. Furthermore, U_f is normalized by T and $(B_f := TU_f,\, N)$ is a BN-pair of G_{fp} with $B_f \cap N = T$. More generally, G_{fp}, U_f, U^-, N, T, $\{\sigma_i \in \mathcal{W} \cong N/T \,|\, i \in I\}$ is a refined Tits system. In particular, there are the *Bruhat* and *Birkhoff decompositions*

$$G_{fp} = \dot{\bigcup_{\sigma\in\mathcal{W}}} B_f\sigma B_f = \dot{\bigcup_{\sigma\in\mathcal{W}}} B^\epsilon\sigma B_f = \dot{\bigcup_{\sigma\in\mathcal{W}}} B_f\sigma B^\epsilon \qquad , \qquad \epsilon \in \{+,-\}.$$

There are Levi decompositions of the standard parabolic subgroups $(P_{fp})_J$, $J \subseteq I$, and of U_f:

$$(P_{fp})_J = (L_{fp})_J \ltimes (U_f)^J \quad \text{and} \quad U_f = (U_f)_J \ltimes (U_f)^J$$

with $(U_f)^J := exp((\mathbf{n}_f)^J)$, $(U_f)_J := exp((\mathbf{n}_f)_J)$ and $(L_{fp})_J = (U_f)_J(\mathcal{W}_J T)(U_f)_J = (U_f)_J L_J = L_J(U_f)_J$, $J \subseteq I$.

The group G_{fp} identifies with the Kac-Moody group of Kapitel 5 in [Sl] for a simply connected minimal free realization. It identifies with the Kac-Moody group of Section 6 in [Ku].

The formal Kac-Moody group G_{fn}: Instead of $\mathcal{O}^p_{int}$ we could have worked with the category $\mathcal{O}^n_{int}$, defined as the full subcategory of the category $\mathcal{O}^n$ whose objects are integrable modules. Every object of $\mathcal{O}^n_{int}$ is isomorphic to a direct sum of integrable irreducible lowest weight modules.

Similarly, we get a monoid $\overset{\vee}{G}$ acting on the objects of $\mathcal{O}^n_{int}$. Its unit group G is isomorphic to the (minimal) Kac-Moody group. We get the formal Kac-moody group $G_{fn} \supseteq G$ acting on the objects of $\mathcal{O}^n_{int}$. It has a refined Tits system $U_f^- := exp(\mathbf{n}_f^-)$, U, N, T, $\{\sigma_i \in \mathcal{W} \cong N/T \,|\, i \in I\}$.

If V is an object of $\mathcal{O}^p_{int}$ resp. $\mathcal{O}^n_{int}$ then the restricted linear dual $V^{(*)}$ equipped with the dual $\mathbf{g}$-action is an object of $\mathcal{O}^n_{int}$ resp. $\mathcal{O}^p_{int}$. The Kac-Moody group G acts on V, $V^{(*)}$ dually. But:

- If V is an object of $\mathcal{O}^p_{int}$ then G_{fp} acts only on $V^* \supseteq V^{(*)}$ dually.
- If V is an object of $\mathcal{O}^n_{int}$ then G_{fn} acts only on $V^* \supseteq V^{(*)}$ dually.

2 Spectra of homogeneous prime ideals of graded algebras

In this section we work out briefly the theory of spectra of homogeneous prime ideals of graded algebras as far as we will need it in this article. Proofs which are variants of the nongraded or $\mathbb{N}_0$-graded case are omitted.

There is the following difference to the projective spectra considered in algebraic geometry. Let $A = \bigoplus_{n\in\mathbb{N}_0} A_n$ be a $\mathbb{N}_0$-graded algebra such that $\mathbb{F}1 \subseteq A_0$. For simplicity assume equalitiy. Denote by $\mathrm{Proj}\,(A)$ all $\mathbb{N}_0$-homogeneous prime ideals of A and by $m := \bigoplus_{n\in\mathbb{N}} A_n$ the irrelevant ideal. Then

$$\mathrm{Proj}\,(A) = \mathrm{Proj}\,(A) \setminus \{m\} \,\dot{\cup}\, \{m\}.$$

Here $\mathrm{Proj}\,(A) \setminus \{m\}$ can be made into a projective scheme as for example explained in [Ha], Chapter II, Section 2. But it is only an open dense part of $\mathrm{Proj}\,(A)$, in the boundary there is the closed point m. The example which we will consider in this article is the projective spectrum $\mathrm{Proj}\,(CA)$ of the Cartan algebra CA. The projective spectrum $\mathrm{Proj}\,(CA)$ is a locally ringed space. There is an open dense part of $\mathrm{Proj}\,(CA)$ which is a scheme, but also its boundary is interesting. Actually, $\mathrm{Proj}\,(CA)$ is stratified by schemes. The algebraic geometric action of the monoid $\widehat{G}$ on $\mathrm{Proj}\,(CA)$, which we will obtain naturally, does not respect this stratification. The existence of such an action is not at all obvious if we would only consider the union of these schemes.

In this section a $\mathcal{M}$-graded algebra A, or graded algebra A for short, consists of the following data:

(a) A commutative monoid $(\mathcal{M}, +)$ with the cancelation property, i.e., for all Λ_1, Λ_2, $N \in \mathcal{M}$ it holds

$$\Lambda_1 + N = \Lambda_2 + N \quad \Rightarrow \quad \Lambda_1 = \Lambda_2.$$

(b) A commutative, associative $\mathbb{F}$-linear algebra $(A, \bullet)$ with unit 1. Furthermore, a $\mathbb{F}$-linear decomposition

$$A = \bigoplus_{\Lambda\in\mathcal{M}} A_\Lambda \quad \text{such that} \quad \mathbb{F}\,1 \subseteq A_0 \quad \text{and} \quad A_\Lambda \bullet A_N \subseteq A_{\Lambda+N} \text{ for all } \Lambda,\, N \in \mathcal{M}.$$

Since the cancelation property holds, the commutative monoid $\mathcal{M}$ embedds into a commutative group $\mathcal{M} - \mathcal{M}$, minimal over $\mathcal{M}$.

Projective spectra and their $\mathbb{F}$-valued points as topological spaces: Let A be a $\mathcal{M}$-graded algebra. We set

$$\mathrm{Proj}\,(A) := \{\mathcal{M}\text{-homogeneous prime ideals of } A\}\,.$$

The closed sets of the Zariski topology of $\mathrm{Proj}\,(A)$ are obtained by

$$\mathcal{V}(I) := \{\, Q \in \mathrm{Proj}\,(A) |\, Q \supseteq I\}\,, \quad I \text{ a } \mathcal{M}\text{-homogeneous ideal of } A.$$

If $Q \in \mathrm{Proj}\,(A)$ then $Q = \bigoplus_{\Lambda\in\mathcal{M}} Q_\Lambda$ with $Q_\Lambda := Q \cap A_\Lambda$. The set $\bigcup_{\Lambda\in\mathcal{M}} A_\Lambda \setminus Q_\Lambda$ of homogeneous elements of $A \setminus Q$ is a multiplicatively closed subset of A. Now

$$A_{(Q)} := \left\{ \frac{a}{b} \middle| \, a \in A_\Lambda,\, b \in A_\Lambda \setminus Q_\Lambda,\, \Lambda \in \mathcal{M} \text{ such that } A_\Lambda \setminus Q_\Lambda \neq \emptyset \right\} \subseteq \left(\bigcup_{\Lambda\in\mathcal{M}} A_\Lambda \setminus Q_\Lambda \right)^{-1} A$$

is a commutative, associative, unital, local $\mathbb{F}$-algebra with maximal ideal

$$m_{(Q)} := \left\{ \frac{a}{b} \middle| \, a \in Q_\Lambda, \, b \in A_\Lambda \setminus Q_\Lambda, \, \Lambda \in \mathcal{M} \text{ such that } A_\Lambda \setminus Q_\Lambda \neq \emptyset \right\}.$$

We define the residue field of Q to be

$$\mathbb{F}_{(Q)} := A_{(Q)}/m_{(Q)}.$$

We identify the field $\mathbb{F}$ with the corresponding subfield of $\mathbb{F}_{(Q)}$.

We call $Q \in \text{Proj}\,(A)$ an $\mathbb{F}$-valued point of $\text{Proj}\,(A)$ if $\mathbb{F}_{(Q)} = \mathbb{F}$. We set

$$\text{Proj}\,(A)(\mathbb{F}) := \left\{ Q \in \text{Proj}\,(A) \middle| \, \mathbb{F}_{(Q)} = \mathbb{F} \right\}.$$

A useful characterization of the $\mathbb{F}$-valued points is given by

Theorem 2.1 *Let $Q \in Proj\,(A)$. It is equivalent:*

(i) $Q \in Proj\,(A)(\mathbb{F})$.

(ii) $dim\, A_\Lambda/Q_\Lambda \leq 1$ for all $\Lambda \in \mathcal{M}$.

Proof: For every $\Lambda \in \mathcal{M}$ we choose a linear complement R_Λ of Q_Λ in A_Λ, i.e., $A_\Lambda = Q_\Lambda \oplus R_\Lambda$. We first show

$$\mathbb{F}_{(Q)} = \left\{ \frac{r}{s} + m_{(Q)} \middle| \, r \in R_\Lambda, \, s \in R_\Lambda \setminus \{0\}, \, \Lambda \in \mathcal{M} \text{ such that } R_\Lambda \neq \{0\} \right\}. \tag{2}$$

Obviously, the inclusion "$\supseteq$" holds. Now let $\frac{a}{b} + m_{(Q)} \in \mathbb{F}_{(Q)}$ where $a \in A_\Lambda$, $b \in A_\Lambda \setminus Q_\Lambda$ and $\Lambda \in \mathcal{M}$ such that $A_\Lambda \setminus Q_\Lambda \neq \emptyset$. Decompose

$$\begin{aligned} a &= q_a + r \qquad \text{with} \qquad q_a \in Q_\Lambda, \; r \in R_\Lambda, \\ b &= q_b + s \qquad \text{with} \qquad q_b \in Q_\Lambda, \; s \in R_\Lambda \setminus \{0\}. \end{aligned}$$

It holds

$$\frac{a}{b} - \frac{r}{s} = \frac{a \bullet s - r \bullet b}{b \bullet s} = \frac{q_a \bullet s - r \bullet q_b}{b \bullet s} \in m_{(Q)},$$

because of $q_a \bullet s$, $r \bullet q_b \in Q_\Lambda \bullet R_\Lambda \subseteq Q_{2\Lambda}$ and $b \bullet s \in (A_\Lambda \setminus Q_\Lambda) \bullet (A_\Lambda \setminus Q_\Lambda) \subseteq A_{2\Lambda} \setminus Q_{2\Lambda}$.

Now suppose that (ii) holds. Let $\Lambda \in \mathcal{M}$ such that $R_\Lambda \neq \{0\}$. If $r \in R_\Lambda$, $s \in R_\Lambda \setminus \{0\}$ then $r = \mu s$ there for some $\mu \in \mathbb{F}$. Therefore, $\frac{r}{s} = \mu \frac{1}{1}$. Now (i) follows by (2).

Suppose that (i) holds. Let $\Lambda \in \mathcal{M}$ such that $R_\Lambda \neq \{0\}$. Choose $s \in R_\Lambda \setminus \{0\}$. By (2) it follows that for every $r \in R_\Lambda$ there exits some $\mu \in \mathbb{F}$ such that

$$\frac{r}{s} - \mu \frac{1}{1} = \frac{r - \mu s}{s} \in m_{(Q)}.$$

Therefore, there exist $a \in Q_N$, $b \in A_N \setminus Q_N$, $N \in \mathcal{M}$ such that

$$\frac{r - \mu s}{s} = \frac{a}{b}.$$

It follows that there exists $c \in A_M \setminus Q_M$, $M \in \mathcal{M}$ such that

$$(r - \mu s) \bullet b \bullet c = a \bullet s \bullet c. \tag{3}$$

Since $a \in Q$ the right hand side of (3) is contained in Q. Assume that $r - \mu s \neq 0$. Since $r - \mu s \in R_\Lambda \setminus \{0\} \subseteq A \setminus Q$ and b, $c \in A \setminus Q$, the left hand side of (3) is contained in $A \setminus Q$, which is not possible. It follows $r = \mu s$. Therefore, (ii) holds. $\square$

The set $\mathrm{Proj}\,(A)(\mathbb{F})$ is equipped with the relative topology of the Zariski topology. The relative closure of $Z \subseteq \mathrm{Proj}\,(A)(\mathbb{F})$ is denoted by $\overline{Z}^{pts}$.

For a subset $S \subseteq \mathrm{Proj}\,(A)$ we define

$$S(\mathbb{F}) := S \cap \mathrm{Proj}\,(A)(\mathbb{F}).$$

Let A be a $\mathcal{M}_A$-graded algebra and B be a $\mathcal{M}_B$-graded algebra. A morphism $f^* : A \to B$ of graded algebras is a unital morphism of algebras $f^* : A \to B$ for which there exists a map $F^* : \mathcal{M}_A \to \mathcal{M}_B$ such that it holds:

(a) $f^*(A_\Lambda) \subseteq B_{F^*(\Lambda)}$ for all $\Lambda \in \mathcal{M}_A$.

(b) F^* is injective on $\{\Lambda \in \mathcal{M}_A |\, f^*(A_\Lambda) \neq \{0\}\}$.

Here, the map F^* is uniquely determined on $\{\Lambda \in \mathcal{M}_A |\, f^*(A_\Lambda) \neq \{0\}\}$. Outside this set F^* is not relevant and can be choosen arbitrary. The definition has been formulated in this way to avoid working with partial maps.

Condition (b) guaranties the following indispensable property: If J is a graded ideal of B then the inverse image $(f^*)^{-1}(J)$ is a graded ideal of A. Now it is not difficult to check

Theorem 2.2 *Let $f^* : A \to B$ be a morphism of graded algebras.*

(a) A continous map is given by

$$\begin{array}{rccc} f: & \mathit{Proj}\,(B) & \to & \mathit{Proj}\,(A) \\ & Q & \to & (f^*)^{-1}(Q) \end{array} .$$

It maps $\mathit{Proj}\,(B)(\mathbb{F})$ into $\mathit{Proj}\,(A)(\mathbb{F})$.

(b) For every $Q \in \mathit{Proj}\,(B)$ the morphism $f^* : A \to B$ induces a morphism of algebras

$$\begin{array}{rccc} f^*_Q: & A_{(f(Q))} & \to & B_{(Q)} \\ & \frac{a}{b} & \to & \frac{f^*(a)}{f^*(b)} \end{array} ,$$

which is local, i.e., $f^*_Q(m_{(f(Q))}) \subseteq m_{(Q)}$. This morphism induces a injective morphism $f^*_Q : \mathbb{F}_{(f(Q))} \to \mathbb{F}_{(Q)}$ of fields over $\mathbb{F}$.

Note that for a morphism $f^* : A \to B$ of graded algebras it is equivalent:

(i) $f^* : A \to B$ is an isomorphism of graded algebras.

(ii) $f^* : A \to B$ is a isomorphism of algebras.

Note also that for a isomorphism $f^* : A \to B$ of graded algebras a map F^* as in the definition maps $\{\Lambda \in \mathcal{M}_A |\, A_\Lambda \neq \{0\}\}$ bijectively to $\{N \in \mathcal{M}_B |\, B_N \neq \{0\}\}$. The monoids $\mathcal{M}_A$ and $\mathcal{M}_B$ may be non-isomorphic, even non-bijective.

Let A be a $\mathcal{M}$-graded algebra. Let $S \subseteq A \setminus \{0\}$ be a multiplicatively closed subset of homogeneous elements. For $\Lambda \in \mathcal{M}$ denote by $S_\Lambda := S \cap A_\Lambda$ the Λ-homogeneous elements of S. Set $\mathcal{M}_S := \{\Lambda \in \mathcal{M} |\, S_\Lambda \neq \emptyset\}$ and $\mathcal{M} - \mathcal{M}_S := \{\Lambda - N \in \mathcal{M} - \mathcal{M} |\, \Lambda \in \mathcal{M},\, N \in \mathcal{M}_S\}$. Set

$$D(S) := \{Q \in \mathrm{Proj}\,(A) |\, S \subseteq A \setminus Q\}.$$

The localization $S^{-1}A$ is a commutative associative unital $(\mathcal{M}-\mathcal{M}_S)$-graded $\mathbb{F}$-algebra. The canonical morphism $i^* : A \to S^{-1}A$ is a morphism of graded algebras. (To guarantie these properties was the reason to demand the cancelation property for $\mathcal{M}$ in our definition of a $\mathcal{M}$-graded algebra A.) It is not difficult to check

Theorem 2.3 *(a) The continous map*

$$\begin{array}{rcl} i : \mathit{Proj}(S^{-1}A) & \to & \mathit{Proj}(A) \\ Q & \mapsto & (i^*)^{-1}(Q) \end{array}$$

maps $\mathit{Proj}(S^{-1}A)$ homeomorphically to $D(S)$. It also maps $\mathit{Proj}(S^{-1}A)(\mathbb{F})$ homeomorphically to $D(S)(\mathbb{F})$.

(b) For every $Q \in \mathit{Proj}(S^{-1}A)$ the local morphism $i^*_Q : A_{(i(Q))} \to (S^{-1}A)_{(Q)}$ is an isomorphism.

In particular, the theorem applies to the principal open sets

$$D(a) := \{ Q \in \mathrm{Proj}(A) | \, a \notin Q \}$$

with $a \in A$ homogeneous and not nilpotent, because of $D(a) = D(\{ a^n | \, n \in \mathbb{N}_0 \})$. These sets give a base of the Zariski topology of $\mathrm{Proj}(A)$.

Let $S \subseteq A \setminus \{0\}$ be a multiplicatively closed subset of homogeneous elements of A. An element $s_0 \in S$ is called principal if it has the following property: For all $s \in S$ there exist $\widetilde{s} \in S$ and $n \in \mathbb{N}_0$ such that $s\widetilde{s} = s_0^n$.

A subset $I \subseteq S$ is called a semigroup ideal of S if $S \bullet I \subseteq I$. Note that we also allow I to be the empty set. A submonoid $F \subseteq S$, for which $S \setminus F$ is a semigroup ideal of S, is called a face of S. The relative interior $ri(S)$ of S is the semigroup ideal of S defined by

$$ri(S) := S \setminus \bigcup_{\substack{F \text{ a face of } S \\ F \neq S}} F.$$

It is not difficult to check

Theorem 2.4 *Suppose there exists a principal element $s_0 \in S$. Then it holds:*

(a) $D(s_0) = D(S)$.

(b) An isomorphism of graded algebras is given by

$$\begin{array}{ccc} \{ s_0^n | \, n \in \mathbb{N}_0 \}^{-1} A & \to & S^{-1}A \\ \frac{a}{s_0^n} & \mapsto & \frac{a}{s_0^n} \end{array}.$$

Furthermore, the set of principal elements of S is a semigroup ideal of S contained in $ri(S)$.

If there exists a principal element of S we call $D(S)$ a principal open set.

Remark 2.5 *If S is generated by finitely many elements $s_1, s_2, \ldots, s_k$ then $s_0 := s_1 s_2 \cdots s_k$ is a principal element of S. But note that finite generation is not necessary for the existence of a principal element.*

Let A be a $\mathcal{M}$-graded algebra. Let I be a graded ideal of A, $I \neq A$. The quotient algebra A/I is $\mathcal{M}$-graded. The canonical morphism $\pi^* : A \to A/I$ is a morphism of graded algebras. It is not difficult to check that it holds

Theorem 2.6 *(a) The continous map*

$$\begin{array}{rcl} \pi : \mathit{Proj}(A/I) & \to & \mathit{Proj}(A) \\ Q & \mapsto & (\pi^*)^{-1}(Q) \end{array}$$

maps $\mathit{Proj}(A/I)$ homeomorphically to $\mathcal{V}(I)$. It also maps $\mathit{Proj}(A/I)(\mathbb{F})$ homeomorphically to $\mathcal{V}(I)(\mathbb{F})$.

(b) For every $Q \in \mathit{Proj}(A/I)$ the local morphism $\pi^*_Q : A_{(\pi(Q))} \to (A/I)_{(Q)}$ is surjective.

If A is a $\mathcal{M}$-graded algebra and B is a $\mathcal{N}$-graded algebra then $A\otimes B$ is a $\mathcal{M}\oplus\mathcal{N}$-graded algebra. The maps

$$i_A^*:\begin{array}{ccc} A & \rightarrow & A\otimes B \\ a & \mapsto & a\otimes 1 \end{array} \quad\text{and}\quad i_B^*:\begin{array}{ccc} B & \rightarrow & A\otimes B \\ b & \mapsto & 1\otimes b \end{array}.$$

are morphisms of graded algebras. Note that in general $(A\otimes B,\, i_A^*,\, i_B^*)$ is no coproduct of A and B in the category of graded algebras. It holds

Theorem 2.7 *(a) The map*

$$\begin{array}{ccc} i_A: Proj(A\otimes B) & \rightarrow & Proj(A) \\ Q & \mapsto & (i_A^*)^{-1}(Q) \end{array}$$

is a continous map, mapping $Proj(A\otimes B)(\mathbb{F})$ into $Proj(A)(\mathbb{F})$.

(b) For every $Q\in Proj(A\otimes B)$ the local morphism $(i_A^*)_Q : A_{(i_A(Q))}\rightarrow (A\otimes B)_{(Q)}$ is injective.

We now consider particular graded algebras to obtain some results on the projective spectra of tensor products, which will be used later.

Let $\mathcal{N}$ be a commutative monoid with cancellation property. Its monoid algebra

$$\mathbb{F}[\mathcal{N}]=\bigoplus_{N\in\mathcal{N}}\mathbb{F}e_N,\quad e_N\bullet e_{N'}=e_{N+N'}\ \text{ for all }\ N,\,N'\in\mathcal{N},$$

over the field $\mathbb{F}$ is a $\mathcal{N}$-graded algebra. The following proposition should be known. We give a proof because we will need soon a variant of it.

Proposition 2.8 *Let C be a commutative associative $\mathbb{F}$-algebra without zero divisors. Let $\mathcal{N}$ be a free commutative group, and let $\mathbb{F}[\mathcal{N}]$ be its group algebra. Then the algebra $C\otimes\mathbb{F}[\mathcal{N}]$ has no zero divisors.*

Remark 2.9 *In particular, for $C=\mathbb{F}$ it follows: The group algebra $\mathbb{F}[\mathcal{N}]$ of a free commutative group $\mathcal{N}$ has no zero divisors.*

Proof: An element $x\in C\otimes\mathbb{F}[\mathcal{N}]$ can be uniquely written in the form $x=\sum_{N\in\mathcal{N}}c_N\otimes e_N$ with $c_N\in C$, $N\in\mathcal{N}$, and $supp(x):=\{N\in\mathcal{N}|\, c_N\neq 0\}$ is finite.

Suppose there exist nonzero elements $x=\sum_{N\in\mathcal{N}}c_N\otimes e_N$, $\widetilde{x}=\sum_{N\in\mathcal{N}}\widetilde{c}_N\otimes e_N$ of $C\otimes\mathbb{F}[\mathcal{N}]$ such that $x\widetilde{x}=0$.

Choose a base of the free commutative group $\mathcal{N}$. There exits a finite subset $\{N_1,\,N_2,\,\ldots N_k\}$ of this base such that $supp(x)$ and $supp(\widetilde{x})$ are contained in the span $\mathbb{Z}N_1\oplus ZN_2\oplus\cdots\oplus\mathbb{Z}N_k$. Order the elements of this span by the lexicographical order, which is a total order, compatible with the addition.

Let M be the maximal element of $supp(x)$ and $\widetilde{M}$ the maximal element of $supp(\widetilde{x})$. Then $M+\widetilde{M}$ is the maximal element of $supp(x)+supp(\widetilde{x})$. Therefore, $x\widetilde{x}$ is of the form

$$x\widetilde{x}=c_M\widetilde{c}_{\widetilde{M}}\otimes e_{M+\widetilde{M}}+\sum_{N<M+\widetilde{M}}d_N\otimes e_N.$$

Since C has no zero divisors it holds $c_M\widetilde{c}_{\widetilde{M}}\neq 0$. This contradicts $x\widetilde{x}=0$. $\square$

The last Proposition is used to show

Theorem 2.10 *Let A be a $\mathcal{M}$-graded algebra. Let $\mathcal{N}$ be a free commutative group, and let $\mathbb{F}[\mathcal{N}]$ be its $\mathcal{N}$-graded group algebra. The map*

$$\begin{array}{ccc} i_A: Proj(A\otimes\mathbb{F}[\mathcal{N}]) & \rightarrow & Proj(A) \\ Q & \mapsto & (i_A^*)^{-1}(Q) \end{array}$$

is a homeomorphism, mapping $Proj(A \otimes \mathbb{F}[\mathcal{N}])(\mathbb{F})$ *bijectively to* $Proj(A)(\mathbb{F})$*. Its inverse is*

$$\begin{array}{rccc} (i_A)^{-1}: & Proj(A) & \to & Proj(A \otimes \mathbb{F}[\mathcal{N}]) \\ & Q & \mapsto & Q \otimes \mathbb{F}[\mathcal{N}] \end{array}.$$

For every $Q \in Proj(A)$ *the local morphism* $(i_A^*)_Q : A_{(Q)} \to (A \otimes \mathbb{F}[\mathcal{N}])_{(Q \otimes \mathbb{F}[\mathcal{N}])}$ *is an isomorphism.*

Remark 2.11 *In particular, for* $A = \mathbb{F}$ *it follows:* $Proj(\mathbb{F}[\mathcal{N}]) = Proj(\mathbb{F}[\mathcal{N}])(\mathbb{F}) = \{\{0\}\}$ *for the group algebra* $\mathbb{F}[\mathcal{N}]$ *of a free commutative group* $\mathcal{N}$*.*

Proof: The map i_A is well defined and continous by Theorem 2.7. If $Q \in \mathrm{Proj}(A)$ then $Q \otimes \mathbb{F}[\mathcal{N}]$ is a graded ideal of $A \otimes \mathbb{F}[\mathcal{N}]$ and it holds

$$(i_A^*)^{-1}(Q \otimes \mathbb{F}[\mathcal{N}]) = \{ a \in A |\, a \otimes 1 \in Q \otimes \mathbb{F}[\mathcal{N}]\} = Q.$$

The ideal $Q \otimes \mathbb{F}[\mathcal{N}]$ is also prime because $(A \otimes \mathbb{F}[\mathcal{N}])/(Q \otimes \mathbb{F}[\mathcal{N}]) \cong (A/Q) \otimes \mathbb{F}[\mathcal{N}]$ has no zero divisors by Proposition 2.8. This shows that the map $(i_A)^{-1}$ given in the theorem is well defined and it holds $i_A \circ (i_A)^{-1} = id$.

Let J be a graded ideal of $A \otimes \mathbb{F}[\mathcal{N}]$. Then $(i_A^*)^{-1}(J)$ is a graded ideal of A, because i_A^* is a graded morphism of algebras. We show that it holds

$$J = (i_A^*)^{-1}(J) \otimes \mathbb{F}[\mathcal{N}]. \tag{4}$$

Since J is an ideal, the inclusion $\supseteq$ holds. Now let $x \in J$. Write x in the form

$$x = \sum_{N \in \mathcal{N}} a_N \otimes e_N \quad \text{with} \quad a_N \in A,\ a_N \neq 0 \text{ for only finitely many } N.$$

Since J is homogeneous it follows $a_N \otimes e_N \in J$ for all $N \in \mathcal{N}$. Since J is an ideal it follows

$$a_N \otimes 1 = (a_N \otimes e_N) \bullet (1 \otimes e_{-N}) \in J \quad \text{for all} \quad N \in \mathcal{N},$$

which is equivalent to $a_N \in (i_A^*)^{-1}(J)$ for all $N \in \mathcal{N}$, which in turn is equivalent to $x \in (i_A^*)^{-1}(J) \otimes \mathbb{F}[\mathcal{N}]$.

From (4) it follows $(i_A)^{-1} \circ i_A = id$. Furthermore, it follows $i_A(\mathcal{V}(J)) = \mathcal{V}((i_A^*)^{-1}(J))$ for every graded ideal J of $A \otimes \mathbb{F}[\mathcal{N}]$, which shows that $(i_A)^{-1}$ is continous.

Let $Q \in \mathrm{Proj}(A)$. By Theorem 2.7 the morphism $(i_A^*)_Q : A_{(Q)} \to (A \otimes \mathbb{F}[\mathcal{N}])_{(Q \otimes \mathbb{F}[\mathcal{N}])}$ is injective. Let $M \in \mathcal{M}$ and $N \in \mathcal{N}$. For $a \otimes e_N \in A_M \otimes e_N$ and $b \in A_M \otimes e_N \setminus Q_M \otimes e_N$ it holds

$$(i_A^*)_Q(\frac{a}{b}) = \frac{a \otimes 1}{b \otimes 1} = \frac{a \otimes e_N}{b \otimes e_N}.$$

Therefore, $(i_A^*)_Q$ is also surjective. It follows that $\mathrm{Proj}(A \otimes \mathbb{F}[\mathcal{N}])(\mathbb{F})$ is mapped bijectively to $\mathrm{Proj}(A)(\mathbb{F})$. □

Theorem 2.12 *Let* A *be a* $\mathcal{M}$*-graded algebra such that* $\mathcal{M} - \mathcal{M}$ *is a free commutative group. Let* B *a* $\mathcal{N}$*-graded algebra such that* $\mathcal{N} - \mathcal{N}$ *is a free commutative group.*

(a) It holds: $\quad Proj(A \otimes B)(\mathbb{F}) \neq \emptyset \iff Proj(A)(\mathbb{F}) \neq \emptyset$ *and* $Proj(B)(\mathbb{F}) \neq \emptyset$*.*

(b) The map

$$\begin{array}{rccc} (i_A,\, i_B): & Proj(A \otimes B)(\mathbb{F}) & \to & Proj(A)(\mathbb{F}) \times Proj(B)(\mathbb{F}) \\ & Q & \mapsto & ((i_A^*)^{-1}(Q),\, (i_B^*)^{-1}(Q)) \end{array}$$

is a bijective continous map. Its inverse is given by

$$\begin{array}{rccc} (i_A,\, i_B)^{-1}: & Proj(A)(\mathbb{F}) \times Proj(B)(\mathbb{F}) & \to & Proj(A \otimes B)(\mathbb{F}) \\ & (R,\, S) & \mapsto & R \otimes B + A \otimes S \end{array}.$$

Proof: Let C be a commutative associative $\mathbb{F}$-algebra without zero divisors. Let $D = \bigoplus_{N\in\mathcal{N}} D_N$ be a $\mathcal{N}$-graded algebra without zero divisors and let $\dim(D_N) \leq 1$ for all $N \in \mathcal{N}$. By adapting the proof of Proposition 2.8 it is possible to show that $C \otimes D$ has no zero divisors.

To (a): If $Q \in \mathrm{Proj}\,(A \otimes B)(\mathbb{F})$ then by Theorem 2.7 it follows $(i_A^*)^{-1}(Q) \in \mathrm{Proj}\,(A)(\mathbb{F})$ and $(i_B^*)^{-1}(Q) \in \mathrm{Proj}\,(B)(\mathbb{F})$.

Now let $R \in \mathrm{Proj}\,(A)(\mathbb{F})$ and $S \in \mathrm{Proj}\,(B)(\mathbb{F})$. Obviously, $R \otimes B + A \otimes S$ is a graded ideal of $A \otimes B$ with homogeneous parts

$$(R \otimes B + A \otimes S)_{(M,\,N)} = R_M \otimes B_N + A_M \otimes S_N \quad \text{for} \quad M \in \mathcal{M},\ N \in \mathcal{N}.$$

It holds

$$\begin{aligned} A \otimes B/(R \otimes B + A \otimes S) &\cong A/R \otimes B/S, \\ (A \otimes B)_{(M,\,N)}/(R \otimes B + A \otimes S)_{(M,\,N)} &\cong A_M/R_M \otimes B_N/S_N \quad \text{for all} \quad M \in \mathcal{M},\ N \in \mathcal{N}. \end{aligned}$$

The algebras A/R and B/S have no zero divisors. Furthermore, it holds $B/S = \bigoplus_{N\in\mathcal{N}} B_N/S_N$ with $\dim(B_N/S_N) \leq 1$ for all $N \in \mathcal{N}$ by Theorem 2.1. It follows that $A/R \otimes B/S$ has no zero divisors. Therefore, $R \otimes B + A \otimes S \in \mathrm{Proj}\,(A \otimes B)$. By Theorem 2.1 it holds $\dim(A_M/R_M \otimes B_N/S_N) = \dim(A_M/R_M)\dim(B_N/S_N) \leq 1$ for all $M \in \mathcal{M}$ and $N \in \mathcal{N}$. By the same theorem it follows $R \otimes B + A \otimes S \in \mathrm{Proj}\,(A \otimes B)(\mathbb{F})$.

To (b): By Theorem 2.7 the map $(i_A,\, i_B)$ is continous. It remains to show that the maps of the theorem are inverse maps. For $R \in \mathrm{Proj}\,(A)(\mathbb{F})$ and $S \in \mathrm{Proj}\,(B)(\mathbb{F})$ it holds

$$(i_A^*)^{-1}(R \otimes B + A \otimes S) = \{\, a \in A |\, a \otimes 1 \in (R \otimes B + A \otimes S)\} = R,$$

because of $1 \notin S$. Similarly, it holds $(i_B^*)^{-1}(R\otimes B + A\otimes S) = S$. Therefore, $(i_A,\, i_B) \circ (i_A,\, i_B)^{-1} = id$.

Now we show $(i_A,\, i_B)^{-1} \circ (i_A,\, i_B) = id$, which is equivalent to

$$i_A(Q) \otimes B + A \otimes i_B(Q) = Q \quad \text{for all} \quad Q \in \mathrm{Proj}\,(A \otimes B)(\mathbb{F}).$$

For all $M \in \mathcal{M}$ and $N \in \mathcal{N}$ it holds

$$i_A(Q)_M \otimes B_N + A_M \otimes i_B(Q)_N \subseteq Q_{M+N} \subseteq A_M \otimes B_N. \tag{5}$$

By Theorem 2.1 it holds $\dim(A_M/i_A(Q)_M) \leq 1$ and $\dim(B_N/i_B(Q)_N) \leq 1$.

If $i_A(Q)_M = A_M$ or $i_B(Q)_N = B_N$ then in (5) both inclusions are equalities. Now suppose that $i_A(Q)_M \neq A_M$, $i_B(Q)_N \neq B_N$, and $i_A(Q)_M \otimes B_N + A_M \otimes i_B(Q)_N \subsetneqq Q_{M+N}$. Then $Q_{M+N} = A_M \otimes B_N$. Let $a \in A_M \setminus i_A(Q)_M$, $b \in B_N \setminus i_B(Q)_N$. Then it holds

$$\underbrace{(a \otimes 1)}_{\notin Q} \bullet \underbrace{(1 \otimes b)}_{\notin Q} = a \otimes b \in A_N \otimes B_M = Q_{N+M},$$

which contradicts that Q is prime. $\square$

The structure sheaves of projective spectra: Let A be a $\mathcal{M}$-graded algebra. We now introduce a locally ringed space $(\mathrm{Proj}\,(A), \mathcal{O}_{Proj(A)})$. We call $\mathcal{O}_{Proj(A)}$ the structure sheaf on $\mathrm{Proj}\,(A)$.

A presheaf $\widetilde{\mathcal{O}}_{Proj(A)}$ of $\mathbb{F}$-algebras on $\mathrm{Proj}\,(A)$ is obtained as follows: For $\emptyset \neq U \subseteq \mathrm{Proj}\,(A)$, U open, let $\widetilde{\mathcal{O}}_{Proj(A)}(U)$ be the commutative, associative, unital $\mathbb{F}$-algebra

$$\widetilde{\mathcal{O}}_{Proj(A)}(U) := \left\{ \frac{a}{b} \,\middle|\, \begin{array}{c} a \in A_\Lambda,\, b \in \bigcap_{Q\in U} A_\Lambda \setminus Q_\Lambda, \\ \Lambda \in \mathcal{M} \text{ such that } \bigcap_{Q\in U} A_\Lambda \setminus Q_\Lambda \neq \emptyset \end{array} \right\} \subseteq \left(\bigcup_{\Lambda\in\mathcal{M}} \bigcap_{Q\in U} A_\Lambda \setminus Q_\Lambda \right)^{-1} A.$$

For $\emptyset \neq U_1 \subseteq U_2 \subseteq \mathrm{Proj}\,(A)$, U_1 and U_2 open, let

$$\begin{array}{cccc} \widetilde{res}_{U_1}^{U_2} : & \widetilde{\mathcal{O}}_{Proj(A)}(U_2) & \to & \widetilde{\mathcal{O}}_{Proj(A)}(U_1) \\ & \frac{a}{b} & \mapsto & \frac{a}{b} \end{array}$$

be the restriction morphism. It is easy to check that the local $\mathbb{F}$-algebra $A_{(Q)}$ is the stalk in $Q \in \mathrm{Proj}\,(A)$, its restriction morphisms given by

$$\begin{array}{rccc} \widetilde{res}^U_Q : & \widetilde{\mathcal{O}}_{Proj(A)}(U) & \to & A_{(Q)} \\ & \frac{a}{b} & \mapsto & \frac{a}{b} \end{array}$$

for $Q \in U \subseteq \mathrm{Proj}\,(A)$, U open.

The structure sheaf $\mathcal{O}_{Proj(A)}$ on $\mathrm{Proj}\,(A)$ is the sheaf associated to the presheaf $\widetilde{\mathcal{O}}_{Proj(A)}$ on $\mathrm{Proj}\,(A)$. The stalk in $Q \in \mathrm{Proj}\,(A)$ identifies with the local $\mathbb{F}$-algebra $A_{(Q)}$.

Let $f^* : A \to B$ be a morphism of graded algebras. We now construct a morphism (f, f^*) of the locally ringed spaces $(\mathrm{Proj}\,(B), \mathcal{O}_{Proj(B)})$ and $(\mathrm{Proj}\,(A), \mathcal{O}_{Proj(A)})$, which we call the morphism induced by $f^* : A \to B$.

For f we take the continous map

$$\begin{array}{rccc} f : & \mathrm{Proj}\,(B) & \to & \mathrm{Proj}\,(A) \\ & Q & \to & (f^*)^{-1}(Q) \end{array} .$$

of Theorem 2.2.

A morphism of presheaves $\widetilde{f}^* : \widetilde{\mathcal{O}}_{Proj(A)} \to f_*(\widetilde{\mathcal{O}}_{Proj(B)})$ is obtained as follows: For $\emptyset \neq U \subseteq \mathrm{Proj}\,(A)$, U open, set

$$\begin{array}{rccc} \widetilde{f}^*(U) : & \widetilde{\mathcal{O}}_{Proj(A)}(U) & \to & \widetilde{\mathcal{O}}_{Proj(B)}(f^{-1}(U)) \\ & \frac{a}{b} & \to & \frac{f^*(a)}{f^*(b)} \end{array} .$$

It is easy to check that for $Q \in \mathrm{Proj}\,(B)$ the morphism induced on the stalks of the structure presheaves is given by the local morphism

$$\begin{array}{rccc} f^*_Q : & A_{(f(Q))} & \to & B_{(Q)} \\ & \frac{a}{b} & \to & \frac{f^*(a)}{f^*(b)} \end{array} .$$

of Theorem 2.2.

This morphism of presheaves induces a morphism of sheaves $f^* : \mathcal{O}_{Proj(A)} \to f_*(\mathcal{O}_{Proj(B)})$ such that the local morphisms $f^*_Q : A_{(f(Q))} \to B_{(Q)}$, $Q \in \mathrm{Proj}\,(B)$, are the induced morphisms on the stalks of the structure sheaves. (Construct the associated sheaves as in the proof of Proposition-Definition 1.2 of [Ha]. Let U be an nonempty open set in $\mathrm{Proj}\,(A)$. Then $f^*(U)$ maps the section $s : U \to \prod_{P\in U} A_{(P)}$ in $\mathcal{O}_{Proj(A)}(U)$ to the section $f^*(U)s : f^{-1}(U) \to \prod_{Q\in f^{-1}(U)} B_{(Q)}$ in $\mathcal{O}_{Proj(B)}(f^{-1}(U))$ obtained by $(f^*(U)s)(Q) := f^*_Q(s(f(Q)))$, $Q \in f^{-1}(U)$.)

From Theorem 2.3 it follows immediately

Corollary 2.13 *Let A be a graded algebra and $a \in A$ be homogeneous, not nilpotent. The canonical morphism $i^* : A \to \{a^n | n \in \mathbb{N}_0\}^{-1}A$ induces an isomorphism of the restriction $(D(a), \mathcal{O}_{Proj(A)}|_{D(a)})$ and $(Proj(\{a^n|n \in \mathbb{N}_0\}^{-1}A), \mathcal{O}_{Proj(\{a^n|n\in\mathbb{N}_0\}^{-1}A)})$.*

From Theorem 2.10 it follows immediately

Corollary 2.14 *Let A be a $\mathcal{M}$-graded algebra. Let $\mathcal{N}$ be a free commutative group, and let $\mathbb{F}[\mathcal{N}]$ be its $\mathcal{N}$-graded group algebra. The canonical map $i^*_A : A \to A \otimes \mathbb{F}[\mathcal{N}]$ induces an isomorphism of $(Proj(A), \mathcal{O}_{Proj(A)})$ and $(Proj(A \otimes \mathbb{F}[\mathcal{N}]), \mathcal{O}_{Proj(A\otimes\mathbb{F}[\mathcal{N}])})$.*

The results on projective spectra of graded algebras obtained so far are sufficient to investigate the projective spectrum $\mathrm{Proj}\,(CA)$ of the Cartan algebra CA associated to a Kac-Moody group. This spectrum has interesting additional structures. We will also see that the locally ringed space $\mathrm{Proj}\,(CA)$ is stratified by schemes. It should be possible to obtain a similar result for the projective spectrum $\mathrm{Proj}\,(A)$ of a $\mathcal{M}$-graded algebra A if its properties, in particular the properties of $\mathcal{M}$, are good enough.

3 The spectrum of homogeneous prime ideals of the Cartan algebra and its $\mathbb{F}$-valued points

The Cartan algebra: We denote by CA the Cartan algebra associated to the Kac-Moody group G, which is a commutative associative unital P^+-graded algebra over the field $\mathbb{F}$ without zero-divisors. It can be obtained by the following well-known construction: Take as P^+-graded $\mathbb{F}$-linear space

$$CA := \bigoplus_{\Lambda \in P^+} L(\Lambda)^{(*)},$$

where $L(\Lambda)^{(*)} := \bigoplus_{\lambda \in P(\Lambda)} L(\Lambda)^*_\lambda \subseteq L(\Lambda)^*$ is the restricted $\mathbb{F}$-linear dual of $L(\Lambda)$. To define the product of CA we fix for every $\Lambda \in P^+$ a highest weight vector $v_\Lambda \in L(\Lambda)_\Lambda \setminus \{0\}$. For $\Lambda,\, N \in P^+$ the G-module $L(\Lambda) \otimes L(N)$ decomposes in the form

$$L(\Lambda) \otimes L(N) = \underbrace{L_{high}}_{\cong L(\Lambda+N)} \oplus \underbrace{L_{low}}_{\substack{\text{only isotypical components} \\ \text{of type } L(M),\, M<\Lambda+N}} .$$

Since $v_\Lambda \otimes v_N$ is a highest weight vector of L_{high}, it is possible to define a G-equivariant linear map

$$\Phi :\, L(\Lambda + N) \to L(\Lambda) \otimes L(N) \quad \text{ by } \quad \Phi(v_{\Lambda+N}) := v_\Lambda \otimes v_N .$$

The product $\bullet$ of the Cartan algebra CA between the parts $L(\Lambda)^{(*)}$, $L(N)^{(*)}$ is now obtained dually to Φ:

$$\bullet : L(\Lambda)^{(*)} \times L(N)^{(*)} \to L(\Lambda)^{(*)} \otimes L(N)^{(*)} \to (L(\Lambda) \otimes L(N))^{(*)} \stackrel{\Phi^{(*)}}{\to} L(\Lambda + N)^{(*)}.$$

Some actions on the Cartan algebra: Let $\Lambda \in P^+$. The face monoid $\widehat{G}$ acts on the highest weight module $L(\Lambda)$. In the obvious way we get an action π of the opposite monoid $\widehat{G}^{op}$ on $L(\Lambda)^{(*)}$.

If we equip $L(\Lambda)^{(*)}$ with its natural $\mathbf{g}$-module structure as a dual of $L(\Lambda)$, it is a lowest weight module of lowest weight $-\Lambda$. The formal Kac-Moody group G_{fn} acts on $L(\Lambda)^{(*)}$. Dually we get an action of G_{fn} on

$$L(\Lambda)_f := (L(\Lambda)^{(*)})^* = \prod_{\lambda \in P(\Lambda)} (L(\Lambda)_\lambda)^{**} = \prod_{\lambda \in P(\Lambda)} L(\Lambda)_\lambda .$$

In this article we do not work with the action of G_{fn} on $L(\Lambda)^{(*)}$, but with the action π of $(G_{fn})^{op}$ on $L(\Lambda)^{(*)}$, which is obtained by concatenation with the inverse map. (This is for the reason that the Kac-Moody group G, which sits inside $\widehat{G}$ and G_{fn}, should act in the same way on $L(\Lambda) \subseteq L(\Lambda)_f$ as well as in the same way on $L(\Lambda)^{(*)}$.)

Consequently, we also do not work with the action of the formal Kac-Moody algebra $\mathbf{g}_{fn} \supseteq \mathbf{g}$ on $L(\Lambda)^{(*)}$, but with the action of $\mathbf{g}_{fn}^{op} \supseteq \mathbf{g}^{op}$ on $L(\Lambda)^{(*)}$, which is obtained by multiplying by -1. For simplicity, we also denote this action by π.

We could have combined the actions of $\widehat{G}^{op}$ and $(G_{fn})^{op}$ on $L(\Lambda)^{(*)}$ to an action of a bigger monoid. For the aims of this article it is not advantageous.

In this way we get an action π of $\widehat{G}^{op}$ as well as an action π of $(G_{fn})^{op}$ on the Cartan algebra CA by morphisms of graded algebras. Recall that for $x \in \widehat{G}$ resp. $x \in G_{fn}$ we get a continous map on the spectrum $\mathrm{Proj}\,(CA)$ of all P^+-homogeneous prime ideals of CA by

$$xQ := \pi(x)^{-1}(Q) \quad \text{ where } \quad Q \in \mathrm{Proj}\,(CA).$$

It leaves the spectrum $\mathrm{Proj}\,(CA)(\mathbb{F})$ of all $\mathbb{F}$-valued points of $\mathrm{Proj}\,(CA)$ invariant. Therefore, we get an action of $\widehat{G}$ as well as an action of G_{fn} on the spectra $\mathrm{Proj}\,(CA)$ and $\mathrm{Proj}\,(CA)(\mathbb{F})$.

The formal building: The Kac-Moody group G has the opposite BN-pairs $(B^\pm,\, N)$ with corresponding standard parabolic subgroups $P_J^\pm$, $J \subseteq I$. The formal Kac-Moody group G_{fn} has

the BN-pair (B_f^-, N) with corresponding standard parabolic subgroups $(P_{fn}^-)_J$, $J \subseteq I$. It holds $(P_{fn}^-)_J \cap G = P_J^-$. Furthermore, for every $\Lambda \in P^+ \cap F_J$ it holds

$$N_{G_{fn}}(L(\Lambda)_\Lambda^*) := \{ g \in G_{fn} | \, gL(\Lambda)_\Lambda^* = L(\Lambda)_\Lambda^* \} = (P_{fn}^-)_J .$$

Now we extend the parabolic subgroups P_J, $J \subseteq I$, of G to subgroups of G_{fn}. For $J \subseteq I$ set

$$(P_{fn})_J := \Big((U_f^-)_J (\mathcal{W}_J T) U_J \Big) U^J = (U_f^-)_J (\mathcal{W}_J T) U = (U_f^-)_J P_J .$$

In particular, $(P_{fn})_\emptyset = B$, $(P_{fn})_{\{i\}} = P_{\{i\}}$ for all $i \in I$, and $(P_{fn})_I = G_{fn}$.

Theorem 3.1 *Let $J \subseteq I$.*

(a) $(P_{fn})_J$ is a subgroup of G_{fn} for which it holds $(P_{fn})_J \cap G = P_J$.

(b) For every $\Lambda \in P^+ \cap F_J$ it holds

$$N_{G_{fn}}(L(\Lambda)_\Lambda) := \{ g \in G_{fn} | \, gL(\Lambda)_\Lambda = L(\Lambda)_\Lambda \} = (P_{fn})_J .$$

Proof: We first show (b). It is well-known that

$$N_G(L(\Lambda)_\Lambda) = P_J \quad \text{ and } \quad N_{\mathbf{g}}(L(\Lambda)_\Lambda) := \{ x \in \mathbf{g} | \, xL(\Lambda)_\Lambda \subseteq L(\Lambda)_\Lambda \} = \mathbf{p}_J := \mathbf{n}_J^- \oplus \mathbf{h} \oplus \mathbf{n}.$$

In particular, it holds $\mathbf{n}_J^- L(\Lambda)_\Lambda \subseteq L(\Lambda)_\Lambda$, from which follows $(\mathbf{n}_f^-)_J L(\Lambda)_\Lambda \subseteq L(\Lambda)_\Lambda$, from which in turn follows $(U_f^-)_J L(\Lambda)_\Lambda = L(\Lambda)_\Lambda$. Therefore, $(P_{fn})_J = (U_f^-)_J P_J \subseteq N_{G_{fn}}(L(\Lambda)_\Lambda)$. Now let $g \in N_{G_{fn}}(L(\Lambda)_\Lambda)$. Decompose g in the form $g = u^J u_J n_\sigma v$ with $u^J \in (U_f^-)^J$, $u_J \in (U_f^-)_J$, $n_\sigma \in N$ projecting to $\sigma \in \mathcal{W}$, and $v \in U^+$. From

$$L(\Lambda)_\Lambda = gL(\Lambda)_\Lambda = u^J u_J n_\sigma L(\Lambda)_\Lambda = u^J u_J L(\Lambda)_{\sigma\Lambda}$$

it follows $\sigma\Lambda = \Lambda$, which is equivalent to $\sigma \in \mathcal{W}_J$, and $u^J L(\Lambda)_\Lambda = L(\Lambda)_\Lambda$. Now u^J acts as

$$u^J = exp(x) \quad \text{ for some } \quad x = \sum_{\alpha \in (\Delta^J)^-} x_\alpha \in \prod_{\alpha \in (\Delta^J)^-} \mathbf{g}_\alpha .$$

Assume that $u^J \neq 1$. Choose a nonzero homogeneous component x_β with β of maximal height. Then $x_\beta v_\Lambda$ is the $(\Lambda + \beta)$-homogeneous part of $u^J v_\Lambda$. It is nonzero because of $x_\beta \notin \mathbf{g}_J + \mathbf{h} = N_{\mathbf{g}}(L(\Lambda)_\Lambda)$. This contradicts $u^J L(\Lambda)_\Lambda = L(\Lambda)_\Lambda$. Therefore, $u^J = 1$ and $g = u_J n_\sigma v \in (P_{fn})_J$.

Now (a) follows by (b) and $N_{G_{fn}}(L(\Lambda)_\Lambda) \cap G = N_G(L(\Lambda)_\Lambda) = P_J$. □

Proposition 3.2 *Let $J, K \subseteq I$. It holds:*

(a) $(P_{fn})_J \subseteq (P_{fn})_K$ if and only if $J \subseteq K$.

(b) $(P_{fn})_J \cap (P_{fn})_K = (P_{fn})_{J \cap K}$.

(c) In general, $(P_{fn})_{J \cup K}$ is not generated by $(P_{fn})_J$ and $(P_{fn})_K$ as a group.

Proof: (a) is easy to check. Use Theorem 3.1 (a), the properties of the standard parabolic subgroups P_L, $L \subseteq I$, of G, and the property $(U_f^-)_J \subseteq (U_f^-)_K$ for $J \subseteq K$.

The inclusion "$\supseteq$" of part (b) follows from part (a). Now let $g \in (P_{fn})_J \cap (P_{fn})_K$. By the Birkhoff decomposition of G_{fn}, and by the definitions of $(P_{fn})_J$, $(P_{fn})_K$, the element g can be written in the forms

$$g = u_J n_\sigma u = \widetilde{u}_K n_\sigma \widetilde{u}$$

with u, $\widetilde{u} \in U$, $n_\sigma \in N$ projecting to $\sigma \in \mathcal{W}_{J\cap K}$ and $u_J \in (U_f^-)_J$, $\widetilde{u}_K \in (U_f^-)_K$. It follows

$$(u_J)^{-1}\widetilde{u}_K \in U_f^- \cap \sigma U \sigma^{-1} = U^- \cap \sigma U \sigma^{-1} = \prod_{\alpha \in \Delta_{re}^- \cap \sigma \Delta_{re}^+} U_\alpha .$$

Since $\sigma \in \mathcal{W}_{J\cap K}$ it holds $\Delta_{re}^- \cap \sigma\Delta_{re}^+ \subseteq (\Delta_{J\cap K})_{re}^-$. Therefore, we get

$$\widetilde{u}_K \in (U_f^-)_K \cap (U_f^-)_J U_{J\cap K}^- = (U_f^-)_K \cap (U_f^-)_J = exp((\mathbf{n}_f^-)_K) \cap exp((\mathbf{n}_f^-)_J).$$

Since the exponential function $exp: \mathbf{n}^- \to U_f^-$ is bijective it holds $exp((\mathbf{n}_f^-)_K \cap (\mathbf{n}_f^-)_J) = exp((\mathbf{n}_f^-)_{J\cap K}) = (U_f^-)_{J\cap K}$. It follows $g = \widetilde{u}_K n_\sigma \widetilde{u} \in (U_f^-)_{J\cap K}(\mathcal{W}_{J\cap K}T)U = (P_{fn})_{J\cap K}$.

Now suppose that for all J, $K \subseteq I$ the group $(P_{fn})_{J\cup K}$ is generated by the groups $(P_{fn})_J$ and $(P_{fn})_K$. Then $G_{fn} = (P_{fn})_I$ is generated by the groups $(P_{fn})_{\{i\}} = P_{\{i\}}$, $i \in I$. It follows $G_{fn} = G$, which is only possible if all components of the generalized Cartan matrix A are of finite type. $\square$

We take as formal building the set

$$\Omega_f := \dot{\bigcup_{J\subseteq I}} G_{fn}/(P_{fn})_J = \{\, g(P_{fn})_J \mid g \in G_{fn},\ J \subseteq I\}$$

partially ordered by the reverse inclusion, i.e., for g, $g' \in G_{fn}$ and J, $J' \subseteq I$,

$$g(P_{fn})_J \leq g'(P_{fn})_{J'} \ :\Longleftrightarrow \ g(P_{fn})_J \supseteq g'(P_{fn})_{J'}.$$

The formal Kac-Moody group G_{fn} acts order preservingly on Ω_f by multiplication from the left. We denote by

$$\mathcal{A}_f := \{\, n(P_{fn})_J \mid n \in N,\ J \subseteq I\}$$

the formal standard apartment of Ω_f. The formal building Ω_f is covered by the formal apartments $g\mathcal{A}_f$, $g \in G_{fn}$.

Remark 3.3 *At first, the construction of a formal building Ω_f by the groups $(P_J)_{fn}$, $J \subseteq I$, may look strange. We give a hand-waving motivation, which shows that it is natural:*

Let $J \subseteq I$. Consider first the classical case, i.e., all components of the generalized Cartan matrix A are of finite type, and take $\mathbb{F} = \mathbb{C}$. Here G/P_J is a well-known manifold. It is coverd by big cells, which are charts of the manifold. The standard big cell $BC(J)$ of G/P_J is obtained as the image of U^- in G/P_J. As a set it is given by

$$BC(J) \cong U^-/(U^- \cap P_J) = U^-/U_J^- \cong (U^-)^J.$$

Now consider the case of an arbitrary generalized Cartan matrix. By analogy, the standard big cell of $G_{fn}/(P_{fn})_J$ is obtained as the image of U_f^- in $G_{fn}/(P_{fn})_J$. As a set it is given by

$$BC(J) \cong U_f^-/(U_f^- \cap (P_{fn})_J) = U_f^-/(U_f^-)_J \cong (U_f^-)^J.$$

In this section we obtain the following results:

- We show that the formal building Ω_f identifies with the $\mathbb{F}$-valued points $\mathrm{Proj}\,(CA)(\mathbb{F})$ of the spectrum $\mathrm{Proj}\,(CA)$ of P^+-homogeneous prime ideals of CA.
- We describe the spectrum $\mathrm{Proj}\,(CA)$ of all P^+-homogeneous prime ideals of CA.

For $\Lambda \in P^+$ let $\delta_\Lambda \in L(\Lambda)^*_\Lambda \subseteq CA$ be defined by $\delta_\Lambda(v_\Lambda) := 1$. It is easy to check that for all Λ, $N \in P^+$ it holds $\delta_\Lambda \bullet \delta_N = \delta_{\Lambda+N}$. Therefore,

$$\widetilde{P^+} := \left\{\, \delta_\Lambda \mid \Lambda \in P^+ \right\}$$

is a multiplicatively closed subset of the Cartan algebra CA, isomorphic to $(P^+, +)$. For $M \subseteq P^+$ we set

$$\widetilde{M} := \{\, \delta_\Lambda \,|\, \Lambda \in M\} \subseteq \widetilde{P^+}.$$

For $\Lambda \in P^+$ we set

$$L(\Lambda)^{(*)}_{\neq\Lambda} := \left\{ \phi \in L(\Lambda)^{(*)} \,\middle|\, \phi(v_\Lambda) = 0 \right\} = \bigoplus_{\lambda \in P(\Lambda)\setminus\{\Lambda\}} L(\Lambda)^*_\lambda \,.$$

Here, as always, a sum over the empty set is defined to be $\{0\}$.

Theorem 3.4 *For $J \subseteq I$ define*

$$P(J) := \bigoplus_{\Lambda \in P^+ \cap \overline{F_J}} L(\Lambda)^{(*)}_{\neq\Lambda} \oplus \bigoplus_{\Lambda \in P^+ \setminus \overline{F_J}} L(\Lambda)^{(*)} \subseteq CA.$$

A G_{fn}-equivariant embedding of the G_{fn}-set Ω_f into the G_{fn}-set $Proj(CA)(\mathbb{F})$ is given by

$$\begin{array}{cccc} \omega: & \Omega_f & \to & Proj(CA)(\mathbb{F}) \\ & g(P_{fn})_J & \mapsto & gP(J) \end{array}.$$

Furthermore, for $g(P_{fn})_J,\, h(P_{fn})_K \in \Omega_f$ it holds

$$g(P_{fn})_J \leq h(P_{fn})_K \qquad \Longleftrightarrow \qquad \overline{gP(J)}^{\,pts} \subseteq \overline{hP(K)}^{\,pts}.$$

Proof: Obviously, $P(J)$ is a P^+-homogeneous linear space different from CA. We first show that $P(J)$ is an ideal. Let $N \in P^+$. If $\Lambda \in P^+ \setminus \overline{F_J}$ then also $N + \Lambda \in P^+ \setminus \overline{F_J}$, because $P^+ \cap \overline{F_J}$ is a face of P^+. It follows

$$L(N)^{(*)} \bullet P(J)_\Lambda = L(N)^{(*)} \bullet L(\Lambda)^{(*)} \subseteq L(N+\Lambda)^{(*)} = P(J)_{N+\Lambda}.$$

Now let $\Lambda \in P^+ \cap \overline{F_J}$. For every $\phi \in L(N)^{(*)}$ and every $\psi \in L(\Lambda)^{(*)}_{\neq\Lambda}$ it holds

$$(\phi \bullet \psi)(v_{N+\Lambda}) = \phi(v_N)\, \underbrace{\psi(v_\Lambda)}_{=0} = 0.$$

It follows

$$L(N)^{(*)} \bullet P(J)_\Lambda = L(N)^{(*)} \bullet L(\Lambda)^{(*)}_{\neq\Lambda} \subseteq L(N+\Lambda)^{(*)}_{\neq\Lambda} \subseteq P(J)_{N+\Lambda}.$$

Next we show that $P(J)$ is prime. $CA/P(J)$ is isomorphic in the obvious way to the monoid algebra $\mathbb{F}\,[\widetilde{P^+ \cap \overline{F_J}}]$, which identifies with the monoid algebra $\mathbb{F}\,[P^+ \cap \overline{F_J}]$, which is a subalgebra of the group algebra $\mathbb{F}\,[P]$ of the lattice P. By Remark 2.9 the algebra $\mathbb{F}\,[P]$ has no zero divisors. Therefore, $CA/P(J)$ has no zero divisors.

It holds $\dim\,(L(\Lambda)^{(*)}/P(J)_\Lambda) \leq 1$ for all $\Lambda \in P^+$. By Theorem 2.1 it follows $P(J) \in \mathrm{Proj}\,(CA)(\mathbb{F})$.

ω is a well-defined G_{fn}-equivariant embedding if and only if for all $J \subseteq I$ it holds

$$\mathrm{Stab}_{G_{fn}}(P(J)) = (P_{fn})_J.$$

Now $g \in \mathrm{Stab}_{G_{fn}}(P(J))$ if and only if the following equationsare satisfied:

$$\pi(g)^{-1} L(\Lambda)^{(*)} = L(\Lambda)^{(*)} \quad \text{ for all } \quad \Lambda \in P^+ \setminus \overline{F_J}, \tag{6}$$

$$\pi(g)^{-1} L(\Lambda)^{(*)}_{\neq\Lambda} = L(\Lambda)^{(*)}_{\neq\Lambda} \quad \text{ for all } \quad \Lambda \in P^+ \cap \overline{F_J}. \tag{7}$$

Equations (6) always hold. Written differently, equations (7) are

$$\left\{ \phi \in L(\Lambda)^{(*)} \,\middle|\, \phi(gv_\Lambda) = 0 \right\} = \bigoplus_{\lambda \in P(\Lambda)\setminus\{\Lambda\}} L(\Lambda)^*_\lambda \quad \text{ for all } \quad \Lambda \in P^+ \cap \overline{F_J},$$

which is equivalent to $\mathbb{F}gv_\Lambda = \mathbb{F}v_\Lambda$ for all $\Lambda \in P^+ \cap \overline{F_J}$. By Theorem 3.1 (b) and Proposition 3.2 (a) this is in turn equivalent to $g \in \bigcap_{K\supseteq J}(P_{fn})_K = (P_{fn})_J$.

Now, $g(P_{fn})_J \leq h(P_{fn})_K$ is by definition equivalent to $g(P_{fn})_J \supseteq h(P_{fn})_K$, which is equivalent $(P_{fn})_J \supseteq g^{-1}h(P_{fn})_K$, which in turn is equivalent to $J \supseteq K$ and $g^{-1}h \in (P_{fn})_J$. On the other hand, it holds $\overline{gP(J)}^{pts} \subseteq \overline{hP(K)}^{pts}$ if and only if $gP(J) \in \overline{hP(K)}^{pts} = \overline{hP(K)} \cap \mathrm{Proj}\,(CA)(\mathbb{F})$ if and only if $gP(J) \supseteq hP(K)$ if and only if $P(J) \supseteq g^{-1}hP(K)$, which is equivalent to the inclusions

$$\bigoplus_{\Lambda\in P^+\setminus\overline{F_J}} L(\Lambda)^{(*)} \quad\supseteq\quad \bigoplus_{\Lambda\in P^+\setminus\overline{F_K}} L(\Lambda)^{(*)}, \tag{8}$$

$$\bigoplus_{\Lambda\in P^+\cap\overline{F_J}} L(\Lambda)^{(*)}_{\neq\Lambda} \quad\supseteq\quad \bigoplus_{\Lambda\in P^+\cap\overline{F_K}\cap\overline{F_J}} \pi(g^{-1}h)^{-1}L(\Lambda)^{(*)}_{\neq\Lambda}. \tag{9}$$

Inclusion (8) is equivalent to $P^+ \setminus \overline{F_J} \supseteq P^+ \setminus \overline{F_K}$, which is equivalent to $\overline{F_J} \subseteq \overline{F_K}$, which in turn is equivalent to $J \supseteq K$. Therefore, inclusion (9) is equivalent to

$$L(\Lambda)^{(*)}_{\neq\Lambda} \supseteq \pi(g^{-1}h)^{-1}L(\Lambda)^{(*)}_{\neq\Lambda} \quad \text{for all} \quad \Lambda \in P^+ \cap \overline{F_J}.$$

Here it holds always equality, because $L(\Lambda)^{(*)}_{\neq\Lambda}$ and $\pi(g^{-1}h)^{-1}L(\Lambda)^{(*)}_{\neq\Lambda} = \pi(h^{-1}g)L(\Lambda)^{(*)}_{\neq\Lambda}$ are 1-codimensional subspaces of $L(\Lambda)^{(*)}$. As we have seen by the calculation of the stabilzers, these equations are equivalent to $g^{-1}h \in (P_{fn})_J$. $\square$

Let $Q = \bigoplus_{\Lambda\in P^+} Q_\Lambda$ be a homogeneous ideal of CA. We say Q has a weight space decomposition if

$$Q_\Lambda = \bigoplus_{\lambda\in P(\Lambda)} (Q_\Lambda)_\lambda \quad \text{with} \quad (Q_\Lambda)_\lambda = Q_\Lambda \cap L(\Lambda)^*_\lambda$$

for all $\Lambda \in P^+$. Equivalently, Q is fixed by the action of the elements of the torus T. The elements of $(Q_\Lambda)_\lambda$ are called elements of weight λ, $\lambda \in P(\Lambda)$, $\Lambda \in P^+$.

Now, the image of the formal standard apartment may be characterized representation theoretically as follows:

Theorem 3.5 *It holds*

$$\omega(\mathcal{A}_f) = \{\, Q \in \omega(\Omega_f) \,|\, Q \textit{ has a weight space decomposition } \}\,.$$

Proof: Obviously, the inclusion "$\subseteq$" holds. Now suppose that $gP(J) \in \omega(\Omega_f)$ has a weight space decomposition. If we write $g \in G_{fn}$ in the form $g = unv$ with $u \in U_f^-$, $n_\sigma \in N$ and $v \in U$ then $gP(J) = unP(J)$. We first show $nP(J) \subseteq unP(J)$. Let $\phi_\lambda \in nP(J)$ be an element of weight λ. Then $\pi(u^{-1})\phi_\lambda = \pi(u)^{-1}\phi_\lambda \in unP(J)$ is of the form

$$\pi(u^{-1})\phi_\lambda = \phi_\lambda + \text{ elements of weights different from } \lambda.$$

Therefore $\phi_\lambda \in unP(J)$. Now $unP(J) \subseteq u^{-1}(unP(J)) = nP(J)$ follows similarly, because also $nP(J)$ has a weight space decomposition. $\square$

It remains to show that the map ω of Theorem 3.4 is surjective, which is not straightforward. It is reached in Corollary 3.24 as a consequence of our description of the full projective spectrum $\mathrm{Proj}\,(CA)$, which we investigate next.

The Cartan algebra CA is a P^+-graded algebra without zero divisors. The faces of P^+ are $P^+\cap\overline{F_J}$, $J \subseteq I$. It follows that for every $J \subseteq I$ there is the semidirect decomposition

$$CA = \bigoplus_{\Lambda\in P^+\cap\overline{F_J}} L(\Lambda)^{(*)} \oplus \bigoplus_{\Lambda\in P^+\setminus\overline{F_J}} L(\Lambda)^{(*)}$$

with graded subalgebra $\bigoplus_{\Lambda\in P^+\cap\overline{F_J}} L(\Lambda)^{(*)}$ and graded prime ideal $\bigoplus_{\Lambda\in P^+\setminus\overline{F_J}} L(\Lambda)^{(*)}$, a sum over the empty set defined to be $\{0\}$.

For every $J\subseteq I$ set

$$\overline{Or(J)} := \mathcal{V}(\bigoplus_{\Lambda\in P^+\setminus\overline{F_J}} L(\Lambda)^{(*)}) = \overline{\{\bigoplus_{\Lambda\in P^+\setminus\overline{F_J}} L(\Lambda)^{(*)}\}},$$

which is a G_{fn}-invariant closed subset of $\mathrm{Proj}\,(CA)$. In particular, it holds $\overline{Or(\emptyset)} = \mathrm{Proj}\,(CA)$ and $\overline{Or(I)} = \{\bigoplus_{\Lambda\in P^+\setminus\overline{F_I}} L(\Lambda)^{(*)}\}$. The following properties are trivial to check:

Proposition 3.6 *Let K, $J\subseteq I$. It holds:*

(a) $\overline{Or(K)}\subseteq\overline{Or(J)}$ *if and only if* $K\supseteq J$.

(b) $\overline{Or(J)}\cap\overline{Or(K)} = \overline{Or(J\cup K)}$.

Proposition 3.7 *Let $J\subseteq I$. The map*

$$\begin{array}{rcl} Proj\,(\bigoplus_{\Lambda\in P^+\cap\overline{F_J}} L(\Lambda)^{(*)}) & \to & \overline{Or(J)} \\ Q & \mapsto & Q\oplus\bigoplus_{\Lambda\in P^+\setminus\overline{F_J}} L(\Lambda)^{(*)} \end{array}$$

is a homeomorphism, mapping $Proj\,(\bigoplus_{\Lambda\in P^+\cap\overline{F_J}} L(\Lambda)^{(*)})(\mathbb{F})$ bijectively to $\overline{Or(J)}(\mathbb{F})$.

Proof: The restriction of the canonical map $CA\to CA/\bigoplus_{\Lambda\in P^+\setminus\overline{F_J}} L(\Lambda)^{(*)}$ to the graded subalgebra $\bigoplus_{\Lambda\in P^+\cap\overline{F_J}} L(\Lambda)^{(*)}$ is an isomorphism of graded algebras. The proposition now follows by Theorem 2.6. $\square$

For every $J\subseteq I$ set

$$Or(J) := \overline{Or(J)}\setminus\bigcup_{I\supseteq K\supsetneqq J}\overline{Or(K)} = \overline{Or(J)}\setminus\bigcup_{i\in I\setminus J}\overline{Or(J\cup\{i\})}.$$

Note that $Or(J)$ is really dense in $\overline{Or(J)}$, because it contains $\bigoplus_{\Lambda\in P^+\setminus\overline{F_J}} L(\Lambda)^{(*)}$. By its definition, $Or(J)$ is open in $\overline{Or(J)}$. Furthermore, it is a G_{fn}-invariant set.

Proposition 3.8 *For $J\subseteq I$ it holds*

$$\overline{Or(J)} = \dot{\bigcup_{I\supseteq K\supseteq J}} Or(K).$$

In particular, it holds $Proj\,(CA) = \dot{\bigcup}_{K\subseteq I} Or(K)$.

Proof: By Proposition 3.6 it holds

$$\overline{Or(J)} = \bigcup_{I\supseteq K\supseteq J}\overline{Or(K)}\supseteq\bigcup_{I\supseteq K\supseteq J} Or(K).$$

Let $Q\in\overline{Or(J)}$. Choose a set $I\supseteq K_{max}\supseteq J$, maximal with respect to the inclusion, such that $Q\in\overline{Or(K_{max})}$. Then $Q\in Or(K_{max})$.

Let K_1, $K_2\subseteq I$ and $K_1\neq K_2$. By Proposition 3.6 it follows

$$Or(K_1)\cap Or(K_2)\subseteq\overline{Or(K_1)}\cap\overline{Or(K_2)}\subseteq\overline{Or(K_1\cup K_2)}. \tag{10}$$

At least one of the sets K_1, K_2 is nonempty. Assume K_2 is nonempty. Then it holds $K_1 \cup K_2 \supsetneqq K_1$. By the definition of $Or(K_1)$ it holds

$$Or(K_1) \cap \overline{Or(K_1 \cup K_2)} = \emptyset. \tag{11}$$

From (10) and (11) it follows

$$Or(K_1) \cap Or(K_2) = Or(K_1) \cap Or(K_2) \cap \overline{Or(K_1 \cup K_2)} = \emptyset.$$

$\square$

Next we describe $Or(J)$ as an intersection of $\overline{Or(J)}$ with an open set in $\mathrm{Proj}\,(CA)$. Later we will describe this open set explicitly as an union of easy principal open sets.

Theorem 3.9 *For $J \subseteq I$ it holds*

$$Or(J) = \overline{Or(J)} \cap \mathit{Proj}\,(CA) \setminus \mathcal{V}(\bigoplus_{\Lambda \in P^+ \cap F_J} L(\Lambda)^{(*)}).$$

Proof: Taking the complement in $\overline{Or(J)}$, the equation of the theorem is equivalent to

$$\bigcup_{K \supsetneqq J} \overline{Or(K)} = \overline{Or(J)} \cap \mathcal{V}(\bigoplus_{\Lambda \in P^+ \cap F_J} L(\Lambda)^{(*)}).$$

Inserting the definition of $\overline{Or(K)}$, $K \supseteq J$, this is equivalent to

$$\bigcup_{K \supsetneqq J} \mathcal{V}(\bigoplus_{\Lambda \in P^+ \setminus \overline{F_K}} L(\Lambda)^{(*)}) = \mathcal{V}(\bigoplus_{\Lambda \in (P^+ \cap F_J) \cup P^+ \setminus \overline{F_J}} L(\Lambda)^{(*)}). \tag{12}$$

The inclusion "$\subseteq$" is valid, because for all $K \supsetneqq J$ it holds

$$P^+ \setminus \overline{F_K} = P^+ \setminus \overline{F_J} \cup (P^+ \cap (\overline{F_J} \setminus \overline{F_K})) \supseteq P^+ \setminus \overline{F_J} \cup (P^+ \cap F_J).$$

Now we show "$\supseteq$". Let Q be in the right hand side of (12). Suppose that Q is not contained in the left hand side of (12). Then for every $K \supsetneqq J$ there exists a weight $\Lambda_K \in P^+ \setminus \overline{F_K}$ and an element $\phi_K \in L(\Lambda_K)^{(*)}$ such that $\phi_K \notin Q$. Since Q is prime it follows

$$Q \not\ni \prod_{K \supsetneqq J} \phi_K \in L(\sum_{K \supsetneqq J} \Lambda_K)^{(*)}. \tag{13}$$

Since $P^+ \setminus \overline{F_K}$ is a semigroup ideal of P^+ it holds

$$\begin{aligned}
\sum_{K \supsetneqq J} \Lambda_K \in \bigcap_{K \supsetneqq J} P^+ \setminus \overline{F_K} \quad &= \quad \bigcap_{K \supsetneqq J} \big(P^+ \setminus \overline{F_J} \cup (P^+ \cap (\overline{F_J} \setminus \overline{F_K}))\big) \\
&= \quad P^+ \setminus \overline{F_J} \cup \bigcap_{K \supsetneqq J} \big(P^+ \cap (\overline{F_J} \setminus \overline{F_K})\big) \\
&= \quad P^+ \setminus \overline{F_J} \cup (P^+ \cap F_J).
\end{aligned}$$

Therefore, (13) contradicts that Q is contained in the right hand side of (12). $\square$

For $J \subseteq I$ define

$$D(J) := \left\{ Q \in \mathrm{Proj}\,(CA) | \, \delta_N \notin Q \text{ for all } N \in P^+ \cap \overline{F_J} \right\}.$$

Proposition 3.10 *Let $J \subseteq I$. The set $D(J)$ is principal open. For every $\Lambda \in P^+ \cap F_J$ it holds*

$$\begin{array}{ccc} D(J) & = & D(\delta_\Lambda), \\ (\widetilde{P^+ \cap \overline{F_J}})^{-1} CA & \cong & \{\delta_\Lambda^n \,|\, n \in \mathbb{N}_0\}^{-1} CA. \end{array}$$

Proof: It is easy to check that $P^+ \cap F_J = ri(P^+ \cap \overline{F_J})$ is the set of principal elements of $P^+ \cap \overline{F_J}$. The proposition now follows from Theorem 2.4. $\square$

The open set of Theorem 3.9 is described in

Theorem 3.11 *Let $J \subseteq I$.*

(a) It holds

$$Proj(CA) \setminus \mathcal{V}(\bigoplus_{\Lambda \in P^+ \cap F_J} L(\Lambda)^{(*)}) = \bigcup_{g \in G_{fn}} gD(J) = \bigcup_{g \in G} gD(J).$$

(b) It holds $(P_{fn}^-)_J\, D(J) = D(J)$. In particular, the unions in (a) can be taken over sets of coset representatives of $G_{fn}/(P_{fn}^-)_J$ resp. G/P_J^-.

Proof: We first show (a). Let $Q \in \mathrm{Proj}\,(CA)$. Then

$$Q \notin \bigcup_{g \in G} gD(J)$$

is equivalent to $\pi(g)Q \notin D(J)$ for all $g \in G$. By Proposition 3.10 this is equivalent to

$$\delta_\Lambda \in \pi(g)Q \quad \text{for all} \quad g \in G \ \text{ and } \ \Lambda \in P^+ \cap F_J. \tag{14}$$

Since $L(\Lambda)^{(*)}$ is an irreducible G^{op}-module, it is spanned by $\pi(G)\delta_\Lambda$. Therefore, (14) is equivalent to

$$Q \supseteq \bigoplus_{\Lambda \in P^+ \cap F_J} L(\Lambda)^{(*)}.$$

These equivalences also hold if G is replaced by G_{fn}.

Let $\Lambda \in P^+ \cap F_J$. By Proposition 3.10 it holds $D(J) = \{Q \in \mathrm{Proj}\,(CA) |\, \delta_\Lambda \notin Q\}$. Now (b) follows from $\pi((P_{fn}^-)_J)\delta_\Lambda = \delta_\Lambda$. $\square$

Let $J \subseteq I$. Since CA has no zero divisors, the canonical map of CA into the localization $(\widetilde{P^+ \cap \overline{F_J}})^{-1}CA$ is injective. We identify CA with the corresponding graded subalgebra of $(\widetilde{P^+ \cap \overline{F_J}})^{-1}CA$. By Theorem 2.3 it holds

Proposition 3.12 *Let $J \subseteq I$. The map*

$$\begin{array}{ccc} Proj((\widetilde{P^+ \cap \overline{F_J}})^{-1}CA) & \to & D(J) \\ Q & \mapsto & Q \cap CA \end{array} \tag{15}$$

is a homeomorphism, mapping $Proj((\widetilde{P^+ \cap \overline{F_J}})^{-1}CA)(\mathbb{F})$ bijectively to $D(J)(\mathbb{F})$.

The next aim is to describe the algebra $(\widetilde{P^+ \cap \overline{F_J}})^{-1}CA$ in a more explict way. It will be reached in Theorem 3.20. We need some preparations before. For $J \subseteq I$ and $\Lambda \in P^+$ set

$$L_J(\Lambda) := \bigoplus_{\lambda \in P_J(\Lambda)} L(\Lambda)_\lambda \ \subseteq \ L(\Lambda) \quad \text{with} \quad P_J(\Lambda) := P(\Lambda) \cap \left(\Lambda - (Q_J)_0^+\right).$$

In particular, $L_\emptyset(\Lambda) = L(\Lambda)_\Lambda$ and $L_I(\Lambda) = L(\Lambda)$.

Proposition 3.13 *Let $\emptyset \neq J \subseteq I$ and $\Lambda \in P^+$. Regard $L(\Lambda)$ as a $\mathbf{g}_J$-module. Then $L_J(\Lambda)$ is a $\mathbf{g}_J$-submodule, which is an irreducible highest weight module of $\mathbf{g}_J$ with highest weight $\Lambda|_{\boldsymbol{h}_J}$ and highest weight space $L(\Lambda)_\Lambda$.*

Proof: Since the author has not found a reference where the proof is given, the nontrivial equality $U(\mathbf{n}_J^-)L(\Lambda)_\Lambda = \bigoplus_{\lambda \in P_J(\Lambda)} L(\Lambda)_\lambda$ is shown. It holds

$$\begin{aligned} L(\Lambda) &= U(\mathbf{n}^-)L(\Lambda)_\Lambda = U((\mathbf{n}^-)^J)U(\mathbf{n}_J^-)L(\Lambda)_\Lambda = U((\mathbf{n}^-)^J)L_J(\Lambda) \\ &= L_J(\Lambda) + \sum_{\beta \in Q^+ \setminus Q_J^+} U((\mathbf{n}^-)^J)_{-\beta}\, L_J(\Lambda). \end{aligned}$$

Here

$$L_J(\Lambda) \quad \subseteq \quad \bigoplus_{\lambda \in P(\Lambda) \cap (\Lambda - (Q_J)_0^+)} L(\Lambda)_\lambda, \tag{16}$$

$$\sum_{\beta \in Q^+ \setminus Q_J^+} U((\mathbf{n}^-)^J)_{-\beta}\, L_J(\Lambda) \quad \subseteq \quad \bigoplus_{\lambda \in P(\Lambda) \cap (\Lambda - (Q_J)_0^+ - Q^+ \setminus Q_J^+)} L(\Lambda)_\lambda. \tag{17}$$

Now $L(\Lambda)$ is the direct sum of the sums in (16) and (17) on the right. It follows that in (16) and (17) it holds equality. $\square$

For $J \subseteq I$ and $\Lambda \in P^+$ set

$$L_J(\Lambda)^{(*)} := \bigoplus_{\lambda \in P_J(\Lambda)} L(\Lambda)^*_\lambda \;\subseteq\; L(\Lambda)^{(*)}.$$

Proposition 3.14 *Let $J \subseteq I$. In CA it holds $L_J(\Lambda)^{(*)} \bullet L_J(N)^{(*)} \subseteq L_J(\Lambda+N)^{(*)}$ for all Λ, $N \in P^+$.*

Proof: Let Λ, $N \in P^+$. From the definition of the Cartan product of CA follows immediately

$$L(\Lambda)^*_\lambda \bullet L(N)^*_\mu \subseteq L(\Lambda+N)^*_{\lambda+\mu} \quad \text{for all} \quad \lambda \in P(\Lambda),\ \mu \in P(N).$$

Since $(\Lambda - (Q_J)_0^+) + (N - (Q_J)_0^+) = (\Lambda + N) - (Q_J)_0^+$, the Proposition now follows from the definitions of $L_J(\Lambda)^{(*)}$, $L_J(N)^{(*)}$, and $L_J(\Lambda+N)^{(*)}$. $\square$

It follows from Proposition 3.13 and Proposition 3.14 that

$$CA_J := \bigoplus_{\Lambda \in P_J^+} L_J(\Lambda)^{(*)}$$

is a P_J^+-graded subalgebra of CA. For $J \neq \emptyset$ it is the Cartan algebra associated to G_J.

Recall that $(U_f^-)^J$ is a prounipotent algebraic group with pronilpotent Lie algebra $(\mathbf{n}_f^-)^J = \prod_{\alpha \in (\Delta^-)^J} \mathbf{g}_\alpha$. Recall that the comorphism of the exponential function $exp : (\mathbf{n}_f^-)^J \to (U_f^-)^J$ gives an isomorphism of the coordinate rings of functions

$$exp^* : \mathbb{F}\,[(U_f^-)^J] \to \mathbb{F}\,[(\mathbf{n}_f^-)^J] = Sym\left(((\mathbf{n}^-)^J)^{(*)}\right)$$

where $((\mathbf{n}^-)^J)^{(*)} := \bigoplus_{\alpha \in (\Delta^J)^-} \mathbf{g}^*_\alpha \subseteq ((\mathbf{n}_f^-)^J)^*$. We equipp $\mathbb{F}\,[(U_f^-)^J]$ with the trivial graduation $\{0\}$.

Corollary 3.15 *Let $J \subseteq I$. It holds*

$$Spec\left(\mathbb{F}\,[(U_f^-)^J]\right)(\mathbb{F}) = \left\{ I(u) |\, u \in (U_f^-)^J \right\},$$

where $I(u)$ denotes the vanishing ideal in $u \in (U_f^-)^J$.

Proof: By the universal property of the symmetric algebra and the linear isomorphisms

$$(\,(\,(\mathbf{n}^-)^J\,)^{(*)}\,)^* = (\bigoplus_{\alpha\in(\Delta^-)^J}\mathbf{g}_\alpha^*\,)^* \cong \prod_{\alpha\in(\Delta^-)^J}(\mathbf{g}_\alpha^*)^* \cong \prod_{\alpha\in(\Delta^-)^J}\mathbf{g}_\alpha = (\mathbf{n}_f^-)^J$$

it follows $\mathrm{Spec}\,(\mathbb{F}\,[(\mathbf{n}_f^-)^J])(\mathbb{F}) = \{I(x)|x\in(\mathbf{n}_f^-)^J\}$, where $I(x)$ is the vanishing ideal in $x\in(\mathbf{n}_f^-)^J$. Since the comorphism $exp^*:\mathbb{F}\,[(U_f^-)^J]\to\mathbb{F}\,[(\mathbf{n}_f^-)^J]$ is an isomorphism of coordinate rings the Corollary holds. $\square$

Recall that the Kac-Moody group G is equipped with a coordinate ring of matrix coefficients such that there holds a Peter-Weyl Theorem: A $G^{op}\times G$-equivariant bijective linear map

$$PW:\ \bigoplus_{\Lambda\in P^+} L(\Lambda)^{(*)}\otimes L(\Lambda)\to\mathbb{F}\,[G]$$

is given by $PW(\phi\otimes v):=\phi(\cdot v)$ for all $\phi\in L(\Lambda)^{(*)}$, $v\in L(\Lambda)$, $\Lambda\in P^+$. Here as usual the matrix coefficient $\phi(\cdot v)$ denotes the function which assigns $g\in G$ the value $\phi(gv)\in\mathbb{F}$. To keep our notation easy we often denote the restrictions of the matrix coefficient $\phi(\cdot v)\in\mathbb{F}\,[G]$ to nonempty subsets of G by the same symbol.

Now $(U^-)^J$ is contained in $(U_f^-)^J$ as well as in G.

Theorem 3.16 *Let $J\subseteq I$. It holds*

(a) $(U^-)^J$ is dense in $(U_f^-)^J$.

(b) The restriction of the coordinate ring $\mathbb{F}\,[(U_f^-)^J]$ to $(U^-)^J$ coincides with the restriction of the coordinate ring $\mathbb{F}\,[G]$ to $(U^-)^J$.

Proof: (b) follows for $U^-\subseteq G_{fn}$ similarly as the corresponding result in Theorem 4.4 and Theorem 4.11 of [M8] for $U^+\subseteq G_{fp}$. In particular it holds

$$\left(\mathbb{F}\,[U_f^-]\,|_{(U_f^-)^J}\right)|_{(U^-)^J} = \mathbb{F}\,[U_f^-]\,|_{(U^-)^J} = \mathbb{F}\,[G]\,|_{(U^-)^J}\ .$$

Here $\mathbb{F}\,[U_f^-]\,|_{(U_f^-)^J} = \mathbb{F}\,[(U_f^-)^J]$ because of $\mathbb{F}\,[\mathbf{n}_f^-]\,|_{(\mathbf{n}_f^-)^J} = \mathbb{F}\,[(\mathbf{n}_f^-)^J]$.

(a) follows for $U^-\subseteq G_{fn}$ also similarly as the corresponding result in Theorem 4.4 and Theorem 4.11 of [M8] for $U^+\subseteq G_{fp}$. (a) follows for $(U^-)^J$ by (b) and Theorem 9 of [M2]. $\square$

For $J\subseteq I$ we set

$$\mathbb{F}\,[(U^-)^J] := \mathbb{F}\,[(U_f^-)^J]\,|_{(U^-)^J} = \mathbb{F}\,[G]\,|_{(U^-)^J}\ .$$

Note that by Theorem 3.16 (a) the coordinate ring $\mathbb{F}\,[(U^-)^J]$ is isomorphic to $\mathbb{F}\,[(U_f^-)^J]$ by the restriction map.

The inverse of the restriction map is described as follows: Let $\phi\in L(\Lambda)^{(*)}$, $v\in L(\Lambda)$, and $\Lambda\in P^+$. By Theorem 9 of [M2] there exists the function $\phi(\cdot v)\in\mathbb{F}\,[(U_f^-)^J]$ which assigns to $u\in(U_f^-)^J$ the value $\phi(uv)\in\mathbb{F}$. It restricts to $\phi(\cdot v)\in\mathbb{F}\,[(U^-)^J]$.

V. Kac and D. Peterson described in Lemma 4.3 of [KP2] the coordinate ring $\mathbb{F}\,[U^-]$ as a symmetric algebra in matrix coefficients. A similar description holds for $\mathbb{F}\,[(U^-)^J]$:

Theorem 3.17 *Let $J\subseteq I$. Let $\Lambda\in P^+\cap F_J$. It holds*

$$\mathbb{F}\,[(U^-)^J] = Sym(\,\left\{\,(\pi(x)\delta_\Lambda)(\cdot v_\Lambda)|\,x\in\boldsymbol{n}^J\right\}\,).$$

Proof: For $x\in\mathbf{n}^J$ define the linear function $f_x:(\mathbf{n}^-)^J\to\mathbb{F}$ by $f_x(y):=(x|y)$. Equipp $(\mathbf{n}^-)^J$ with the coordinate ring

$$\mathbb{F}\,[(\mathbf{n}^-)^J] := Sym(\bigoplus_{\alpha\in(\Delta^-)^J}\mathbf{g}_\alpha^*\,) = Sym(\,\left\{\,f_x|\,x\in\mathbf{n}^J\right\}\,).$$

Set $h := \nu^{-1}(\Lambda) \in \mathbf{h}$. In the proof of Theorem 8 of [M2] we showed that the map

$$\begin{array}{rccc} \psi_h : & (U^-)^J & \to & (\mathbf{n}^-)^J \\ & u & \mapsto & Ad(u)h - h \end{array}$$

is well defined, its comorphism $\psi_h^* : \mathbb{F}\left[(\mathbf{n}^-)^J\right] \to \mathbb{F}\left[(U^-)^J\right]$ exists and is surjective. Also, for all $u \in (U^-)^J$ it holds

$$\begin{aligned} &\psi_h^*(f_x)(u) = (x \mid Ad(u)h - h) = (x \mid Ad(u)h) = (Ad(u^{-1})x \mid h) = ((Ad(u^{-1})x)_0 \mid h) \\ &= \Lambda((Ad(u^{-1})x)_0) = \delta_\Lambda((Ad(u^{-1})x)v_\Lambda) = \delta_\Lambda(u^{-1}(x(uv_\Lambda))) = (\pi(x)(\pi(u^{-1})\delta_\Lambda))(uv_\Lambda) \\ &= (\pi(x)\delta_\Lambda)(uv_\Lambda), \end{aligned}$$

where $(Ad(u^{-1})x)_0$ denotes the zero-weight component in $\mathbf{h} = \mathbf{g}_0$ of $Ad(u^{-1})x \in \mathbf{g}$.

For the proof of these results, Lemma 4.3 of [KP2] has been used and also some ideas of the proof of Lemma 4.3 of [KP2]. It remains to show that ψ_h^* is injectve. We do this by generalizing the corresponding part of the proof of Lemma 4.3 of [KP2] for $\mathbb{F}[U]$ to $\mathbb{F}\left[(U^-)^J\right]$. We use the unipotent group U_f^- to make it easier to understand.

Let $(U_f^-)_j$, $j \in \mathbb{N}$ be the descending central series of U_f^-, i.e.,

$$(U_f^-)_1 := U_f^- \quad \text{ and } \quad (U_f^-)_{j+1} := ((U_f^-)_j,\, U_f^-),\ j \in \mathbb{N}.$$

Note that

$$\mathbf{n}_f^- = \prod_{j \in \mathbb{N}} \mathbf{g}_{-j} \quad \text{ with } \quad \mathbf{g}_{-j} := \bigoplus_{\substack{\alpha \in \Delta^- \\ ht(\alpha) = -j}} \mathbf{g}_\alpha$$

Every $u \in (U_f^-)_j$ is of the form

$$u = exp(\phi(u)) \quad \text{ with } \quad \phi(u) = \sum_{\substack{k \in \mathbb{N} \\ k \geq j}} \phi(u)_k, \ \ \phi(u)_k \in \mathbf{g}_{-k}$$

Let $j,\, j' \in \mathbb{N}$. It holds:

(a) $\phi_j(uu') = \phi_j(u) + \phi_j(u')$ for all $u,\, u' \in U_j^-$.

(b) $\phi_{j+j'}((u,\, u')) = [\phi_j(u),\, \phi_{j'}(u')]$ for all $u \in U_j^-$, $u' \in U_{j'}^-$.

Now define

$$(U^-)_1^J := (U^-)^J \quad \text{ and } \quad (U^-)_{j+1}^J := ((U^-)_j^J,\, U^-),\ j \in \mathbb{N}.$$

It holds $(U^-)_j^J \subseteq (U_f^-)_j$ for all $j \in \mathbb{N}$. We first show

$$\phi_j((U^-)_j^J) = (\mathbf{n}^-)^J \cap \mathbf{g}_{-j} \tag{18}$$

by induction over $j \in \mathbb{N}$.

For all $\alpha \in \Delta_{re}^-$ it holds

$$\phi_1(U_\alpha) = \begin{cases} \mathbf{g}_\alpha & \text{if } \alpha = -\alpha_i \text{ for some } i \in I, \\ \{0\} & \text{else.} \end{cases} \tag{19}$$

The group $(U^-)^J$ is the normalizer of U^- generated by the root groups U_α, $\alpha \in (\Delta^-)_{re}^J$. From (19) and property (a) it follows $\phi_1((U^-)^J) = \bigoplus_{i \in I \setminus J} \mathbf{g}_{\alpha_i} = (\mathbf{n}^-)^J \cap \mathbf{g}_{-1}$.

Now suppose it holds $\phi_j((U^-)_j^J) = (\mathbf{n}^-)^J \cap \mathbf{g}_{-j}$ for some $j \in \mathbb{N}$. The group U^- is generated by the root groups U_α, $\alpha \in \Delta_{re}^-$. From (19) and property (a) it follows $\phi_1(U^-) = \mathbf{g}_{-1}$. By property (b) it follows

$$\phi_{j+1}((U^-)_{j+1}^J) = \phi_{j+1}(((U^-)_j^J,\, U^-)) = [\phi_j((U^-)_j^J),\, \phi_1(U^-)] = [(\mathbf{n}^-)^J \cap \mathbf{g}_{-j},\, \mathbf{g}_{-1}].$$

Since $(\mathbf{n}^-)^J$ is an ideal of $\mathbf{n}^-$ it holds $[(\mathbf{n}^-)^J \cap \mathbf{g}_{-j},\, \mathbf{g}_{-1}] \subseteq (\mathbf{n}^-)^J \cap \mathbf{g}_{-j-1}$. Furthermore, by [K], §1.3, it holds

$$(\mathbf{n}^-)^J \cap \mathbf{g}_{-j-1} = \bigoplus_{\substack{\alpha\in(\Delta^J)^- \\ ht(\alpha)=-j-1}} \mathbf{g}_\alpha$$

$$\text{with} \quad \mathbf{g}_\alpha = span\left\{ [\cdots[[f_{i_1},\, f_{i_2}],\, \ldots],\, f_{i_{j+1}}] \,\middle|\, \begin{array}{c} i_1,\, i_2,\, \ldots,\, i_{j+1} \in I, \\ -(\alpha_{i_1} + \alpha_{i_2} + \cdots \alpha_{i_{j+1}}) = \alpha \end{array} \right\}$$

Consider a multibracket $mb := [\cdots[[f_{i_1},\, f_{i_2}],\, \ldots],\, f_{i_{j+1}}]$ such that $-(\alpha_{i_1} + \alpha_{i_2} + \cdots \alpha_{i_{j+1}}) = \alpha \in (\Delta^-)^J$. Then at least one $\alpha_{i_k} \in (\Delta^-)^J$. If $k \neq j+1$ then $mb \in [(\mathbf{n}^-)^J \cap \mathbf{g}_{-j},\, \mathbf{g}_{-1}]$ because $(\mathbf{n}^-)^J$ is an ideal of $\mathbf{n}^-$. If $k = j+1$ then mb can be written as a sum of elements of $[(\mathbf{n}^-)^J \cap \mathbf{g}_{-j},\, \mathbf{g}_{-1}]$ by the Jacobi identity.

Next we show

$$\psi_h((U^-)^J) \equiv (\mathbf{n}^-)^J \mod (\mathbf{n}^-)^J \cap \bigoplus_{\substack{k\in\mathbb{N} \\ k\geq j}} \mathbf{g}_{-k} \tag{20}$$

by induction over $j \in \mathbb{N}$.

Trivially, (20) holds for $j = 1$. Now suppose (20) holds for some $j \in \mathbb{N}$. Let $x \in (\mathbf{n}^-)^J \cap \bigoplus_{k=1}^j \mathbf{g}_{-k}$. Write x as a sum $x = x_{<j} + x_j$ such that $x_{<j} \in (\mathbf{n}^-)^J \cap \bigoplus_{k=1}^{j-1} \mathbf{g}_{-k}$ and $x_j \in (\mathbf{n}^-)^J \cap \mathbf{g}_{-j}$. By the induction assumption there exists an element $u \in (U^-)^J$ such that

$$\begin{aligned} \psi_h(u) = Ad(u)h - u \quad &\equiv \quad x_{<j} \mod (\mathbf{n}^-)^J \cap \bigoplus_{\substack{k\in\mathbb{N} \\ k\geq j}} \mathbf{g}_{-k} \\ &\equiv \quad x_{<j} + r_j \mod (\mathbf{n}^-)^J \cap \bigoplus_{\substack{k\in\mathbb{N} \\ k\geq j+1}} \mathbf{g}_{-k} \end{aligned}$$

with $r_j \in (\mathbf{n}^-)^J \cap \mathbf{g}_{-j}$. Now for every $\widetilde{u} \in (U^-)^J_j$ it follows

$$\begin{aligned} Ad(\widetilde{u}u)h - h \quad &= \quad Ad(\widetilde{u})(Ad(u)h - h) + (Ad(\widetilde{u})h - h) \\ &\equiv \quad x_{<j} + r_j + [\phi_j(\widetilde{u}),\, h] \mod (\mathbf{n}^-)^J \cap \bigoplus_{\substack{k\in\mathbb{N} \\ k\geq j+1}} \mathbf{g}_{-k}. \end{aligned}$$

It is easy to check that it holds $\alpha(h) = (\alpha \mid \Lambda) \neq 0$ for all $\alpha \in (\Delta^-)^J$, compare the proof of Theorem 8 in [M2]. Furthermore, by (18) it holds $\phi_j((U^-)^J) = (\mathbf{n}^-)^J \cap \mathbf{g}_{-j}$. Therefore, there exists an element $\widetilde{u} \in (U^-)^J$ such that $[\phi_j(\widetilde{u}),\, h] + r_j = x_j$.

Now we show that the map $\psi_h^* : \mathbb{F}\,[(\mathbf{n}^-)^J] \to \mathbb{F}\,[(U^-)^J]$ is injective. Let $f \in \mathbb{F}\,[(\mathbf{n}^-)^J]$ such that $\psi_h^*(f) = 0$. Write f in the form

$$f = c1 + \sum_{k=1}^{p} \prod_{l=1}^{q(k)} f_{x_{kl}} \quad \text{with} \quad c \in \mathbb{F},\ x_{kl} \in \mathbf{n}^J.$$

Let $y \in (\mathbf{n}^-)^J$. Choose $j \in \mathbb{N}$ such that $x_{kl} \in (\mathbf{n}^-)^J \cap \bigoplus_{m=1}^{j-1} \mathbf{g}_{-m}$ for all $k,\, l$. Write y as a sum $y = y_{<j} + y_{\geq j}$ with $y_{<j} \in (\mathbf{n}^-)^J \cap \bigoplus_{m=1}^{j-1} \mathbf{g}_{-m}$ and $y_{\geq j} \in (\mathbf{n}^-)^J \cap \bigoplus_{\substack{m\in\mathbb{N} \\ m\geq j}} \mathbf{g}_{-m}$. By (20) there exists an element $u \in (U^-)^J$ such that

$$\psi_h(u) = y_{<j} + r_{\geq j} \quad \text{with} \quad r_{\geq j} \in (\mathbf{n}^-)^J \cap \bigoplus_{\substack{m\in\mathbb{N} \\ m\geq j}} \mathbf{g}_{-m}.$$

It holds

$$\begin{aligned} f(y) \quad &= \quad c + \sum_{k=1}^{p} \prod_{l=1}^{q(k)} (x_{kl} \mid y) = c + \sum_{k=1}^{p} \prod_{l=1}^{q(k)} (x_{kl} \mid \psi_h(u) - r_{\geq j} + y_{\geq j}) \\ &= \quad c + \sum_{k=1}^{p} \prod_{l=1}^{q(k)} (x_{kl} \mid \psi_h(u)) = \psi_h^*(f)(u) = 0. \end{aligned}$$

Since $y \in (\mathbf{n}^-)^J$ was arbitrary it follows $f = 0$. $\square$

We will use the following Corollary, which is not as strong as the last theorem.

Corollary 3.18 *Let $J \subseteq I$. It holds*

$$\mathbb{F}\left[(U^-)^J\right] = span\left\{ \phi(\cdot v_N) \,\middle|\, \phi \in L(N)^{(*)},\ N \in P^+ \cap \overline{F_J} \right\}.$$

Proof: Trivially, it holds

$$\mathbb{F}\left[(U^-)^J\right] \supseteq \text{span}\left\{ \phi(\cdot v_N) \,\middle|\, \phi \in L(N)^{(*)},\ N \in P^+ \cap \overline{F_J} \right\} \supseteq \left\{ (\pi(x)\delta_\Lambda)(\cdot v_\Lambda) \,\middle|\, x \in \mathbf{n}^J \right\}.$$

Since the algebra $\mathbb{F}\left[(U^-)^J\right]$ is generated by $\left\{ (\pi(x)\delta_\Lambda)(\cdot v_\Lambda) \,\middle|\, x \in \mathbf{n}^J \right\}$ it is sufficient to show that this span is a subalgebra of $\mathbb{F}\left[(U^-)^J\right]$. It contains $\delta_0(\cdot v_0)$, which is the unit of $\mathbb{F}\left[(U^-)^J\right]$. Now let $\phi \in L(N)^{(*)}$, $N \in P^+ \cap \overline{F_J}$ and $\psi \in L(M)^{(*)}$, $M \in P^+ \cap \overline{F_J}$. It holds

$$\phi(\cdot v_N)\psi(\cdot v_M) = (\phi \otimes \psi)(\cdot(v_N \otimes v_M)) = (\phi \bullet \psi)(\cdot v_{N+M})$$

with $\phi \bullet \psi \in L(N+M)^{(*)}$ and $N + M \in P^+ \cap \overline{F_J}$. $\square$

We will also use

Proposition 3.19 *Let $J \subseteq I$. Let $\phi \in L_J(\Lambda)^{(*)}$, $v \in L_J(\Lambda)$, $\Lambda \in P^+$. It holds*

$$\phi(uv) = \phi(v) \quad \textit{for all} \quad u \in (U_f^-)^J.$$

Proof: It holds $\phi \in \bigoplus_{\lambda \in \Lambda - (Q_J)_0^+} L(\Lambda)_\lambda^*$ and $v \in \bigoplus_{\lambda \in \Lambda - (Q_J)_0^+} L(\Lambda)_\lambda$. Furthermore, $u \in (U_f^-)^J$ acts on v as $exp(x)$ for some $x \in \prod_{\alpha \in (\Delta^J)^-} \mathbf{g}_\alpha$. It follows

$$uv - v \in \prod_{\lambda \in \Lambda - (Q_J)_0^+ - Q^+ \setminus (Q_J)^+} L(\Lambda)_\lambda.$$

Therefore, $\phi(uv - v) = 0$. $\square$

For $J \subseteq I$ let

$$\mathbb{F}\left[P \cap (\overline{F_J} - \overline{F_J})\right] = \bigoplus_{\Lambda \in P \cap (\overline{F_J} - \overline{F_J})} \mathbb{F} e_\Lambda$$

be the group algebra of the lattice $P \cap (\overline{F_J} - \overline{F_J})$, equipped with its natural $P \cap (\overline{F_J} - \overline{F_J})$-graduation.

Theorem 3.20 *Let $J \subseteq I$. There exists an isomorphism of graded algebras*

$$\Gamma : (\widetilde{P^+ \cap \overline{F_J}})^{-1} CA \to \mathbb{F}\left[(U_f^-)^J\right] \otimes CA_J \otimes \mathbb{F}\left[P \cap (\overline{F_J} - \overline{F_J})\right]$$

such that

$$\begin{array}{lcll} \Gamma(\frac{\delta_N}{\delta_\Lambda}) & = & 1 \otimes 1 \otimes e_{N-\Lambda} & \textit{for all} \quad N, \Lambda \in P^+ \cap \overline{F_J}, \\ \Gamma(\phi) & = & 1 \otimes \phi \otimes 1 & \textit{for all} \quad \phi \in L_J(N)^{(*)},\ N \in P_J^+, \\ \Gamma(\frac{\phi}{\delta_N}) & = & \phi(\cdot v_N) \otimes 1 \otimes 1 & \textit{for all} \quad \phi \in L(N)^{(*)},\ N \in P^+ \cap \overline{F_J}. \end{array}$$

Proof: We denote by $\mathbb{F}[G]^U$ the functions of $\mathbb{F}[G]$ which take constant values on the elements of every coset gU, $g \in G$. Recall that there holds a Borel-Weyl Theorem: An isomorphism of algebras

$$BW : \ CA = \bigoplus_{\Lambda \in P^+} L(\Lambda)^{(*)} \to \mathbb{F}[G]^U$$

is given by $BW(\phi) := \phi(\cdot v_\Lambda)$ for all $\phi \in L(\Lambda)^{(*)}$, $\Lambda \in P^+$. We equipp G/U with the coordinate ring of functions $\mathbb{F}[G/U]$ given in the obvious way by $\mathbb{F}[G]^U$. Trivially, $\mathbb{F}[G/U]$ is isomorphic to $\mathbb{F}[G]^U$.

For $\Lambda \in P^+$ set

$$\theta_\Lambda := BW(\delta_\Lambda) = \delta_\Lambda(\cdot v_\Lambda) \in \mathbb{F}[G]^U .$$

For $J \subseteq I$ define

$$D_{G/U}(J) := \left\{ gU \in G/U \,|\, \theta_\Lambda(gU) \neq 0 \text{ for all } \Lambda \in P^+ \cap \overline{F_J} \right\} .$$

Equipp this set with the coordinate ring of functions

$$\mathbb{F}[D_{G/U}(J)] := \left\{ \frac{f}{\theta_\Lambda}|_{D_{G/U}(J)} \,\middle|\, f \in \mathbb{F}[G]^U ,\ \Lambda \in P^+ \cap \overline{F_J} \right\} .$$

An isomorphism of algebras

$$BW^J : \widetilde{(P^+ \cap \overline{F_J})}^{-1} CA \to \mathbb{F}[D_{G/U}(J)] \tag{21}$$

is given by $BW^J(\frac{\phi}{\delta_\Lambda}) := \frac{BW(\phi)}{\theta_\Lambda}|_{D_{G/U}(J)}$ for all $\phi \in CA$, $\Lambda \in P^+ \cap \overline{F_J}$. The injectivity holds, because $D_{G/U}(J)$ is dense in G/U, which can be seen as follows: The big cell U^-TU is dense in G. It follows that U^-TU/U and also $D_{G/U}(J) \supseteq U^-TU/U$ are dense in G/U.

For $J \subseteq I$ we denote by $\mathbb{F}[G_J]$ the restriction of $\mathbb{F}[G]$ to G_J. Similar as for $\mathbb{F}[G]$ there holds a Peter-Weyl Theorem: A $G_J^{op} \times G_J$-equivariant bijective linear map

$$PW_J : \bigoplus_{\Lambda \in P_J^+} L_J(\Lambda)^{(*)} \otimes L_J(\Lambda) \to \mathbb{F}[G_J]$$

is given by $PW_J(\phi \otimes v) := \phi(\cdot v)$ for all $\phi \in L_J(\Lambda)^{(*)}$, $v \in L_J(\Lambda)$, $\Lambda \in P_J^+$.

We denote by $\mathbb{F}[G_J]^{U_J}$ the functions of $\mathbb{F}[G_J]$ which take constant values on the elements of every coset gU_J, $g \in G_J$. Similar as for $\mathbb{F}[G]^U$ there holds a Borel-Weyl Theorem: An isomorphism of algebras

$$BW_J : CA_J = \bigoplus_{\Lambda \in P_J^+} L_J(\Lambda)^{(*)} \to \mathbb{F}[G_J]^{U_J} \tag{22}$$

is given by $BW_J(\phi) := \phi(\cdot v_\Lambda)$ for all $\phi \in L_J(\Lambda)^{(*)}$, $\Lambda \in P_J^+$. We equipp G_J/U_J with the coordinate ring of functions $\mathbb{F}[G_J/U_J]$ given by $\mathbb{F}[G_J]^{U_J}$. It is isomorphic to $\mathbb{F}[G_J]^{U_J}$.

For $J \subseteq I$ the group algebra $\mathbb{F}[P \cap (\overline{F_J} - \overline{F_J})]$ identifies with the classical coordinate ring $\mathbb{F}[T^J]$ of the torus T^J.

Recall from Corollary 3.18 that for $J \subseteq I$ it holds

$$\mathbb{F}[(U^-)^J] = \operatorname{span}\left\{ \phi(\cdot v_N) \,|\, \phi \in L(N)^{(*)},\ N \in P^+ \cap \overline{F_J} \right\} .$$

Now we show: The map

$$\begin{array}{rcl} m : (U^-)^J \times G_J/U_J \times T^J & \to & D_{G/U}(J) \\ (u^-,\, gU_J,\, t) & \mapsto & u^- gtU \end{array}$$

is well defined and induces a comorphism

$$m^* : \mathbb{F}[D_{G/U}(J)] \to \mathbb{F}[(U^-)^J] \otimes \mathbb{F}[G_J/U_J] \otimes \mathbb{F}[T^J],$$

which is an isomorphims of algebras. Furthermore, it holds:

$$m^*\left(\frac{\theta_N}{\theta_\Lambda}|_{D_{G/U}(J)}\right) = 1\otimes 1\otimes e_{N-\Lambda} \quad \text{for} \quad N,\,\Lambda\in P^+\cap\overline{F_J}, \tag{23}$$

$$m^*\left(\phi(\cdot v_N)|_{D_{G/U}(J)}\right) = 1\otimes \phi(\cdot v_N)|_{G_J/U_J}\otimes 1 \quad \text{for} \quad \phi\in L_J(N)^{(*)},\ N\in P_J^+. \tag{24}$$

$$m^*\left(\frac{\phi(\cdot v_N)}{\theta_N}|_{D_{G/U}(J)}\right) = \phi(\cdot v_N)|_{(U^-)^J}\otimes 1\otimes 1 \quad \text{for} \quad \phi\in L(N)^{(*)},\ N\in P^+\cap\overline{F_J}. \tag{25}$$

We first show that m is well defined as a map to G/U. Let $u^-\in(U^-)^J$, $gU_J=g'U_J\in G_J/U_J$, and $t\in T^J$. Then it holds $u^-gtU=u^-g'tU$ if and only if $gtUt^{-1}=g'tUt^{-1}$ if and only if $gU=g'U$, which follows from $gU_J=g'U_J$.

Next we show that m is a surjective map onto $D_{G/U}(J)$. Let $gU\in G/U$. Write gU in the form $gU=u^-n_\sigma U$ with $u^-\in U^-$, $n_\sigma\in N$ projecting to $\sigma\in\mathcal{W}$. Then it holds $gU\in D_{G/U}(J)$ if and only if

$$\delta_\Lambda(u^-n_\sigma v_\Lambda)\neq 0 \quad \text{for all} \quad \Lambda\in P^+\cap\overline{F_J},$$

if and only if $\sigma\Lambda=\Lambda$ for all $\Lambda\in P^+\cap\overline{F_J}$, if and only if $\sigma\in\mathcal{W}_J$. Therefore,

$$D_{G/U}(J)=U^-N_JT^J/U=(U^-)^JU_J^-N_JT^J/U=m((U^-)^J,\,G_J/U_J,\,T^J).$$

Now let $\phi\in L(N)^*_\lambda$, $\lambda\in P(N)$, and $N\in P^+$. Let $\Lambda\in P^+\cap\overline{F_J}$. Note that $G_J\subseteq N_G(L(\Lambda)_\Lambda)$. Decompose $N=N_J+N^J$ with $N_J\in P_J^+$ and $N^J\in P^+\cap\overline{F_J}$. Choose dual bases $\phi_{\mu\,j}\in L(N)^*_\mu$, $v_{\mu\,j}\in L(N)_\mu$, $\mu\in P(N)$, $j=1,\ldots,\dim(L(N)_\mu)$. For all $u^-\in(U^-)^J$, $g\in G_J$, and $t\in T^J$ it holds

$$m^*\left(\frac{\phi(\cdot v_N)}{\theta_\Lambda}|_{D_{G/U}(J)}\right)(u^-,\,gU,\,t)=\frac{\phi(u^-gtv_N)}{\theta_\Lambda(u^-gt)}=\phi(u^-gv_N)e_{N^J-\Lambda}(t) \tag{26}$$

$$=\sum_{\mu,\,j}\phi(u^-v_{\mu\,j})\phi_{\mu\,j}(gv_N)e_{N^J-\Lambda}(t)$$

$$=\left(\sum_{\substack{\mu,\,j\\ \mu\geq\lambda}}\phi(\cdot v_{\mu\,j})\otimes\phi_{\mu\,j}(\cdot v_N)\otimes e_{N^J-\Lambda}\right)(u^-,\,gU,\,t). \tag{27}$$

The sum in (27) is finite. It follows that the comorphism m^* exists. Again by $G_J\subseteq N_G(L(N)_N)$ for all $N\in P^+\cap\overline{F_J}$ it follows (23) and (25) from (26). By Proposition 3.19 it follows (24) from (26). In particular, m^* is surjective. Furthermore, the comorphism m^* is injective, because m is surjective.

The coordinate ring $\mathbb{F}[(U_f^-)^J]$ is isomorphic to $\mathbb{F}[(U^-)^J]$ by the restriction map whose inverse has been described above. There are the isomorphisms (21) and (22). By these isomorphisms, the isomorphism m^* coincides with Γ.

The algebra $\widetilde{(P^+\cap\overline{F_J})}^{-1}CA$ is graded by

$$P^+-(P^+\cap\overline{F_J})=P\cap(\overline{C}-\overline{F_J}).$$

The algebra $\mathbb{F}[(U_f^-)^J]\otimes CA_J\otimes\mathbb{F}[P\cap(\overline{F_J}-\overline{F_J})]$ is graded by

$$\{0\}+P_J^+ +(P\cap(\overline{F_J}-\overline{F_J}))=P\cap(\overline{C}-\overline{F_J}).$$

The isomorphism Γ is an isomorphism of graded algebras because it preserves the grading on homogeneous generators of the algebras. $\square$

For $J\subseteq I$ we call

$$BC(J):=\overline{Or(J)}\cap D(J)\subseteq Or(J)$$

the standard big cell of $Or(J)$. We call every $gBC(J)$, $g\in G_{fn}$, a big cell of $Or(J)$.

Corollary 3.21 *For $J \subseteq I$ it holds:*

(a) Every big cell $gBC(J)$, $g \in G_{fn}$, is principal open in $\overline{Or(J)}$ and dense in $\overline{Or(J)}$.

(b) $Or(J)$ can be coverd by big cells:

$$Or(J) = \bigcup_{g \in G_{fn}} gBC(J) = \bigcup_{g \in G} gBC(J).$$

It holds $(P_{fn})_J^- \, BC(J) = BC(J)$. In particular, the unions can be taken over sets of coset representatives of $G_{fn}/(P_{fn})_J^-$ resp. G/P_J^-.

Proof: Part (b) follows immeditately from Theorem 3.9 and Theorem 3.11. It is sufficient to show part (a) for $BC(J)$ because $\overline{Or(J)}$ is G_{fn}-invariant. $BC(J)$ is principal open in $\overline{Or(J)}$ because $D(J)$ is principal open in $\mathrm{Proj}\,(CA)$ by Proposition 3.10. By the definition of $\overline{Or(J)}$ and $BC(J)$ it follows

$$\bigoplus_{\Lambda \in P^+ \setminus \overline{F_J}} L(\Lambda)^{(*)} \in BC(J) = \overline{Or(J)} \cap D(J) \subseteq \overline{Or(J)} = \overline{\{ \bigoplus_{\Lambda \in P^+ \setminus \overline{F_J}} L(\Lambda)^{(*)} \}}.$$

Therefore, $BC(J)$ is dense in $\overline{Or(J)}$. $\square$

Corollary 3.22 *Let $J \subseteq I$. It holds:*

(a) The map

$$\begin{array}{ccc} \mathit{Proj}\,((\widetilde{P^+ \cap \overline{F_J}})^{-1} \bigoplus_{\Lambda \in P^+ \cap \overline{F_J}} L(\Lambda)^{(*)}) & \to & BC(J) \\ Q & \mapsto & (Q \cap \bigoplus_{\Lambda \in P^+ \cap \overline{F_J}} L(\Lambda)^{(*)}) \oplus \bigoplus_{\Lambda \in P^+ \setminus \overline{F_J}} L(\Lambda)^{(*)} \end{array}$$

is a homeomorphism, mapping $\mathit{Proj}\,((\widetilde{P^+ \cap \overline{F_J}})^{-1} \bigoplus_{\Lambda \in P^+ \cap \overline{F_J}} L(\Lambda)^{(*)})(\mathbb{F})$ bijectively to $BC(J)(\mathbb{F})$.

(b) There exists an isomorphism of graded algebras

$$\Gamma : (\widetilde{P^+ \cap \overline{F_J}})^{-1} \bigoplus_{\Lambda \in P^+ \cap \overline{F_J}} L(\Lambda)^{(*)} \to \mathbb{F}\,[(U_f^-)^J] \otimes \mathbb{F}\,[P \cap (\overline{F_J} - \overline{F_J})]$$

such that

$$\begin{array}{rclll} \Gamma(\frac{\delta_N}{\delta_\Lambda}) & = & 1 \otimes e_{N-\Lambda} & \textit{for all} & N, \Lambda \in P^+ \cap \overline{F_J}, \\ \Gamma(\frac{\phi}{\delta_N}) & = & \phi(\cdot v_N) \otimes 1 & \textit{for all} & \phi \in L(N)^{(*)},\ N \in P^+ \cap \overline{F_J}. \end{array}$$

Proof: Part (a) follows from Proposition 3.7, the definition of $D(J)$, and Theorem 2.3. The isomorphism of graded algebras of part (b) is obtained by restricting the graded isomorphim of Theorem 3.20 to $(\widetilde{P^+ \cap \overline{F_J}})^{-1} \bigoplus_{\Lambda \in P^+ \cap \overline{F_J}} L(\Lambda)^{(*)}$. $\square$

Now we come back to the $\mathbb{F}$-valued points of $\mathrm{Proj}\,(CA)$.

Theorem 3.23 *Let $J \subseteq I$. It holds:*

(a) $BC(J)(\mathbb{F}) = (U_f^-)^J P(J)$.

(b) $Or(J)(\mathbb{F}) = G_{fn}\, P(J)$.

Proof: We first show (a). From Corollary 3.22, Theorem 2.10, and Theorem 3.15 it follows that the map

$$\begin{array}{rcl}\beta:\,(U_f^-)^J & \to & BC(J)(\mathbb{F}) \\ u & \mapsto & \left(\Gamma^{-1}\left(I(u)\otimes\mathbb{F}\left[P\cap(\overline{F_J}-\overline{F_J})\right]\right)\cap\bigoplus_{\Lambda\in P^+\cap\overline{F_J}}L(\Lambda)^{(*)}\right)\oplus\bigoplus_{\Lambda\in P^+\setminus\overline{F_J}}L(\Lambda)^{(*)}\end{array}$$

is bijective. For $u\in(U_f^-)^J$ we now compute $\beta(u)$ explicitly. Let $\phi\in L(\Lambda)^{(*)}$, $\Lambda\in P^+\cap\overline{F_J}$. Then it holds

$$\phi\in\Gamma^{-1}\left(I(u)\otimes\mathbb{F}\left[P\cap(\overline{F_J}-\overline{F_J})\right]\right)$$

if and only if

$$\Gamma(\frac{\phi}{\delta_0})=\phi(\cdot v_\Lambda)\otimes e_\Lambda\in I(u)\otimes\mathbb{F}\left[P\cap(\overline{F_J}-\overline{F_J})\right]$$

if and only if

$$0=\phi(uv_\Lambda)=(\pi(u)\phi)(v_\Lambda)$$

if and only if $\pi(u)\phi\in L(\Lambda)^{(*)}_{\neq\Lambda}$. Since $\Gamma^{-1}\left(I(u)\otimes\mathbb{F}\left[P\cap(\overline{F_J}-\overline{F_J})\right]\right)$ is graded we have shown

$$\beta(u)=\bigoplus_{\Lambda\in P^+\cap\overline{F_J}}\pi(u)^{-1}L(\Lambda)^{(*)}_{\neq\Lambda}\oplus\bigoplus_{\Lambda\in P^+\setminus\overline{F_J}}L(\Lambda)^{(*)}=uP(J).$$

From (a) and Corollary 3.21 (b) it follows

$$Or(J)(\mathbb{F})=\bigcup_{g\in G_{fn}}gBC(J)(\mathbb{F})=G_{fn}(U_f^-)^JP(J)=G_{fn}\,P(J).$$

□

Now finally we have reached

Corollary 3.24 *The map ω of Theorem 3.4 is surjective, i.e., it holds*

$$\mathit{Proj}(CA)(\mathbb{F})=\{\,gP(J)|\,g\in G_{fn},\,J\subseteq I\}\,.$$

Proof: By Proposition 3.8 and Theorem 3.23 (b) it follows

$$\mathrm{Proj}\,(CA)(\mathbb{F})=\bigcup_{J\subseteq I}Or(J)(\mathbb{F})=\{\,gP(J)|\,g\in G_{fn},\,J\subseteq I\}\,.$$

□

For $J\subseteq I$ identify the topological space $\overline{Or(J)}$ with $\mathit{Proj}(\bigoplus_{\Lambda\in P^+\cap\overline{F_J}}L(\Lambda)^{(*)})$ by Proposition 3.7. By this identification $\overline{Or(J)}$ gets the structure of a locally ringed space of $\mathbb{F}$-algebras.

Corollary 3.25 *Let $J\subseteq I$. The open set $Or(J)$ of $\overline{Or(J)}$ equipped with its locally ringed substruture is a scheme.*

Proof: Follows from Corollary 3.21, Corollary 3.22, and Corollary 2.14. □

4 An action of the face monoid on its building

Now we use the results of the last section to obtain an algebraic geometric model of an action of the face monoid $\widehat{G}$ on the building of its unit group, the Kac-Moody group G.

We take as building the set

$$\Omega := \dot{\bigcup_{J\subseteq I}} G/P_J = \{ gP_J \,|\, g \in G,\ J \subseteq I\}$$

partially ordered by the reverse inclusion, i.e., for g, $g' \in G$ and J, $J' \subseteq I$,

$$gP_J \leq g'P_{J'} \ :\Longleftrightarrow\ gP_J \supseteq g'P_{J'}.$$

The Kac-Moody group G acts order preservingly on Ω by multiplication from the left. We denote by

$$\mathcal{A} := \{ nP_J \,|\, n \in N,\ J \subseteq I\}$$

the standard apartment of Ω. The building Ω is covered by the apartments $g\mathcal{A}$, $g \in G$.

The building Ω is a substructure of the formal building Ω_f:

Proposition 4.1 *We get an order preserving, G-equivariant embedding by*

$$\begin{array}{cccc} j: & \Omega & \to & \Omega_f \\ & gP_J & \mapsto & g(P_{fn})_J \end{array}.$$

The standard apartment $\mathcal{A}$ is mapped bijectively to the formal standard apartment $\mathcal{A}_f$.

Proof: This follows immediately from $(P_{fn})_J \cap G = P_J$, $J \subseteq I$, which has been shown in Theorem 3.1 (a). $\square$

Recall that for $J \subseteq I$ we set

$$P(J) := \bigoplus_{\Lambda\in P^+\cap\overline{F_J}} L(\Lambda)^{(*)}_{\neq\Lambda} \oplus \bigoplus_{\Lambda\in P^+\setminus\overline{F_J}} L(\Lambda)^{(*)} \subseteq CA$$

with $L(\Lambda)^{(*)}_{\neq\Lambda} := \bigoplus_{\lambda\in P(\Lambda)\setminus\{\Lambda\}} L(\Lambda)^*_\lambda$, a sum over the empty set defined to be $\{0\}$.

Corollary 4.2 *We get a G-equivariant embedding of the G-set Ω into the G-set Proj$(CA)(\mathbb{F})$ by*

$$\begin{array}{cccc} \omega: & \Omega & \to & \mathit{Proj}(CA)(\mathbb{F}) \\ & gP_J & \mapsto & gP(J) \end{array}.$$

Its image is dense in Proj$(CA)(\mathbb{F})$. Furthermore, for gP_J, $hP_K \in \Omega$ it holds

$$gP_J \leq hP_K \quad\Longleftrightarrow\quad \overline{gP(J)}^{pts} \cap \omega(\Omega) \subseteq \overline{hP(K)}^{pts} \cap \omega(\Omega).$$

Proof: The existence of the G-equivariant embedding $\omega : \Omega \to \mathrm{Proj}\,(CA)(\mathbb{F})$ follows from Theorem 3.4 and Proposition 4.1.

For $\omega(\Omega)$ to be dense in $\mathrm{Proj}\,(CA)(\mathbb{F})$ it is sufficient that the G-orbit of $P(\emptyset)$ is dense in $\mathrm{Proj}\,(CA)$. Since $P(\emptyset)$ is P^+-homogeneous and G acts by P^+-homogeneous morphisms it holds

$$\begin{aligned}
\bigcap_{g\in G} gP(\emptyset) &= \bigoplus_{\Lambda\in P^+} \Big(\bigcap_{g\in G} \pi(g)^{-1} L(\Lambda)^{(*)}_{\neq\Lambda}\Big) \\
&= \bigoplus_{\Lambda\in P^+} \Big\{ \phi \in L(\Lambda)^{(*)} \,\Big|\, \pi(g)\phi \in L(\Lambda)^{(*)}_{\neq\Lambda} \text{ for all } g \in G \Big\} \\
&= \bigoplus_{\Lambda\in P^+} \Big\{ \phi \in L(\Lambda)^{(*)} \,\Big|\, \phi(gv_\Lambda) = 0 \text{ for all } g \in G \Big\}.
\end{aligned}$$

For every $\Lambda \in P^+$ the orbit Gv_Λ spans $L(\Lambda)$, because $L(\Lambda)$ is an irreducible G-module. It follows $\bigcap_{g\in G} gP(\emptyset) = \{0\}$. Therefore, $\overline{GP(\emptyset)} = \mathcal{V}(\{0\}) = \mathrm{Proj}\,(CA)$.

By Theorem 3.4 and Proposition 4.1 it holds $gP_J \leq hP_K$ if and only if

$$\overline{gP(J)}^{pts} \subseteq \overline{hP(K)}^{pts}, \tag{28}$$

from which it follows

$$\overline{gP(J)}^{pts} \cap \omega(\Omega) \subseteq \overline{hP(K)}^{pts} \cap \omega(\Omega). \tag{29}$$

Now suppose that (29) holds. Because of

$$gP(J) \in \overline{gP(J)}^{pts} \cap \omega(\Omega) \subseteq \overline{hP(K)}^{pts} \cap \omega(\Omega) \subseteq \overline{hP(K)}^{pts}.$$

it follows (28). $\square$

The face monoid $\widehat{G}$ acts on $\mathrm{Proj}\,(CA)(\mathbb{F})$. It holds

Theorem 4.3 *The image $\omega(\Omega)$ is a $\widehat{G}$-invariant subset of Proj$(CA)(\mathbb{F})$, the action of $\widehat{G}$ on $\omega(\Omega)$ obtained as follows: Let $g_1 e(R(\Theta))g_2 \in \widehat{G}$ and $hP(J) \in \omega(\Omega)$. Decompose $g_2 h$ in the form*

$$g_2 h = a n_y c$$

with $a \in P^-_{\Theta\cup\Theta^\perp}$, $c \in P_J$, and $n_y \in N$ projecting to $y \in {}^{\Theta\cup\Theta^\perp}\mathcal{W}^J$. Then it holds

$$(g_1 e(R(\Theta))g_2)\, hP(J) = g_1 p^-_\Theta(a) P(\Theta \cup J \cup red\,(y)). \tag{30}$$

If we identify the building Ω with its image $\omega(\Omega)$, then this action coincides with good action 1 of $\widehat{G}$ on Ω given in Corollary 50 of [M5].

Proof: It remains to show formula (30). By Theorem 3.4 and by Theorem 9 (c) of [M5] it holds

$$(g_1 e(R(\Theta))g_2)hP(J) = g_1 e(R(\Theta)) a n_y c P(J) = g_1 p^-_\Theta(a) e(R(\Theta)) n_y P(J).$$

Therefore, we only have to show

$$e(R(\Theta)) n_y P(J) = P(\Theta \cup J \cup red\,(y)). \tag{31}$$

Let $\Lambda \in P^+ \setminus \overline{F_J}$. Trivially, it holds

$$\pi(e(R(\Theta))n_y)^{-1} P(J)_\Lambda = \pi(e(R(\Theta))n_y)^{-1} L(\Lambda)^{(*)} = L(\Lambda)^{(*)}. \tag{32}$$

Let $\Lambda \in P^+ \cap \overline{F_J}$. Then it holds

$$\begin{aligned}
\pi(e(R(\Theta))n_y)^{-1} P(J)_\Lambda &= \pi(e(R(\Theta))n_y)^{-1} L(\Lambda)^{(*)}_{\neq\Lambda} \\
&= \left\{ \phi \in L(\Lambda)^{(*)} \,\middle|\, \pi(e(R(\Theta))n_y)\phi \in L(\Lambda)^{(*)}_{\neq\Lambda} \right\} \\
&= \left\{ \phi \in L(\Lambda)^{(*)} \,\middle|\, \phi(e(R(\Theta))n_y v_\Lambda) = 0 \right\} \\
&= \begin{cases} \left\{ \phi \in L(\Lambda)^{(*)} \,\middle|\, \phi(n_y v_\Lambda) = 0 \right\} & if \quad y\Lambda \in R(\Theta) \\ \left\{ \phi \in L(\Lambda)^{(*)} \,\middle|\, \phi(0) = 0 \right\} & if \quad y\Lambda \notin R(\Theta) \end{cases} \\
&= \begin{cases} \left\{ \phi \in L(\Lambda)^{(*)} \,\middle|\, \phi(n_y v_\Lambda) = 0 \right\} & if \quad y\Lambda \in R(\Theta) \\ L(\Lambda)^{(*)} & if \quad y\Lambda \notin R(\Theta) \end{cases} .
\end{aligned}$$

By Theorem 7 of [M5] it holds $R(\Theta) \cap y\overline{F_J} = \overline{F_{\Theta\cup J\cup red(y)}}$. Note also that y fixes the points of $\overline{F_{\Theta\cup J\cup red(y)}}$. It follows

$$y\Lambda \in R(\Theta) \iff y\Lambda \in \overline{F_{\Theta\cup J\cup red(y)}} \iff \Lambda \in \overline{F_{\Theta\cup J\cup red(y)}}.$$

Furthermore, for $\Lambda \in \overline{F_{\Theta \cup J \cup red(y)}}$ we get $0 \neq n_y v_\Lambda \in L(\Lambda)_{y\Lambda} = L(\Lambda)_\Lambda$. Therefore, for $\Lambda \in P^+ \cap \overline{F_J}$, we have shown

$$\pi(e(R(\Theta))n_y)^{-1}P(J)_\Lambda = \begin{cases} L(\Lambda)^{(*)}_{\neq\Lambda} & if \quad \Lambda \in \overline{F_{\Theta\cup J\cup red(y)}} \\ L(\Lambda)^{(*)} & if \quad \Lambda \notin \overline{F_{\Theta\cup J\cup red(y)}} \end{cases} . \tag{33}$$

Since $\overline{F_{\Theta\cup J\cup red(y)}} \subseteq \overline{F_J}$, from (32) and (33) it follows (31). $\square$

Some open problems:

(a) To define the Cartan algebra CA the restricted duals $L(\Lambda)^{(*)}$, $\Lambda \in P^+$, have been used. Similarly, it is possible to obtain a Cartan algebra CA_f, the restricted duals replaced by the full duals $L(\Lambda)^*$, $\Lambda \in P^+$. Investigate Proj (CA_f)! Does it hold Proj $(CA_f)(\mathbb{F}) \cong \dot{\bigcup}_{J\subseteq I} G_{fp}/(P_{fp})_J$ where now really the parabolic subgroups $(P_{fp})_J$, $J \subseteq I$, of G_{fp} enter?

(b) A Cartan algebra CA_M can be obtained for every normal reductive algebraic monoid M. Investigate Proj (CA_M) and the action of M on Proj (CA_M)!

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